

# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



### THESIS

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#### STABILITY ANALYSIS OF SHIP STEERING IN CANALS

by

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March, 1997

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19980109 100

| REPORT DOCUMENTATION PAGE  |  |   | Form Approved OMB No. 0704-0188                |   |
|--|--|---|--|---|
| Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instruction, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington DC 20503.                                    |  |   |  |   |
| 1. AGENCY USE ONLY (Leave blank)   |  | 2. REPORT DATE<br>March 1997                            |  | 3. REPORT TYPE AND DATES COVERED<br>Master's Thesis |
| 4. TITLE AND SUBTITLE<br>STABILITY ANALYSIS OF SHIP STEERING IN CANALS   |  |   | 5. FUNDING NUMBERS                             |   |
| 6. AUTHOR(S)<br>Panos E. Kapasakis   |  |   |  |   |
| 7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)<br>Naval Postgraduate School<br>Monterey CA 93943-5000  |  |   | 8. PERFORMING ORGANIZATION REPORT NUMBER       |   |
| 9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)  |  |   | 10. SPONSORING/MONITORING AGENCY REPORT NUMBER |   |
| 11. SUPPLEMENTARY NOTES<br>The views expressed in this thesis are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.  |  |   |  |   |
| 12a. DISTRIBUTION/AVAILABILITY STATEMENT<br>Approved for public release; distribution is unlimited.  |  |   | 12b. DISTRIBUTION CODE                         |   |
| 13. ABSTRACT (maximum 200 words)<br>The problem of ship steering in canals and confined waters is analyzed with emphasis on stability and bifurcation analysis. The classical maneuvering equations of motion augmented with a model for ship/canal interaction are used to model the open loop dynamics. Coupling of a control law and a guidance scheme with appropriate time lags is employed to model the essential dynamics of a helmsman. The complete system is analyzed using both linear and nonlinear techniques in order to assess its stability under finite disturbances. The results indicate that for certain regions of parameters, limit cycle oscillations may develop which could compromise system stability and safety of operations. |  |   |  |   |
| 14. SUBJECT TERMS<br>SHIP STEERING BIFURCATIONS LIMIT CYCLES   |  |   | 15. NUMBER OF PAGES<br>94                      |   |
|  |  |   | 16. PRICE CODE                                 |   |
| 17. SECURITY CLASSIFICATION OF REPORT<br>Unclassified  | 18. SECURITY CLASSIFICATION OF THIS PAGE<br>Unclassified | 19. SECURITY CLASSIFICATION OF ABSTRACT<br>Unclassified | 20. LIMITATION OF ABSTRACT<br>UL               |   |



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**STABILITY ANALYSIS OF SHIP STEERING IN CANALS**

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Submitted in partial fulfillment  
of the requirements for the degree of

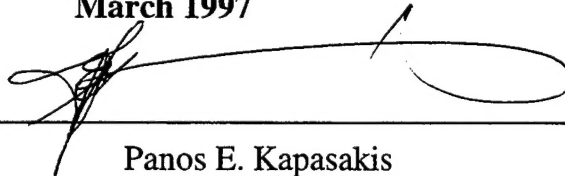
**MASTER OF SCIENCE IN MECHANICAL ENGINEERING**

from the

**NAVAL POSTGRADUATE SCHOOL**

**March 1997**

Author:



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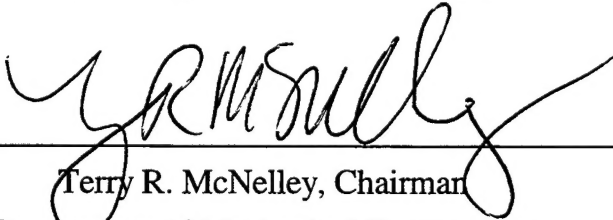
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## ABSTRACT

The problem of ship steering in canals and confined waters is analyzed with emphasis on stability and bifurcation analysis. The classical maneuvering equations of motion augmented with a model for ship/canal interaction are used to model the open loop dynamics. Coupling of a control law and a guidance scheme with appropriate time lags is employed to model the essential dynamics of a helmsman. The complete system is analyzed using both linear and nonlinear techniques in order to assess its stability under finite disturbances. The results indicate that for certain regions of parameters, limit cycle oscillations may develop which could compromise system stability and safety of operations.



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## **I. INTRODUCTION**

### **A. PROBLEM STATEMENT**

The problem of motion stability of ships and other marine vehicles has been the subject of extensive studies in the past (Comstock, 1977). Most of these studies are with regards the ship's directional stability in open waters under open loop conditions. It is well known that in restricted waters such as canals or rivers, although it is possible to have positional stability in principle, in reality it is never the case. This is due to the destabilizing effects of the bank suction forces and moments. These act always in a destabilizing fashion; i.e., after a small initial disturbance they produce a force or a moment which tends to increase the ship's path deviation from nominal (Comstock, 1977). Open loop conditions, important as they are, rarely represent real life applications. Ships traveling in canals are under closed loop control, typically provided through the helmsman or an autopilot. It becomes then important to assess the stability of the system under finite disturbances and incorporating some modeling of closed loop operations.

### **B. OBJECTIVES AND OUTLINE**

This thesis utilizes both linear and nonlinear techniques in order to analyze the problem of closed loop dynamic stability of ships in canals. Ship modeling

is discussed in Chapter II and it follows standard ship maneuvering equations (Clayton and Bishop, 1982) augmented by a model for bank suction effects (Beck, 1976). A Mariner class ship is selected as a model because of the availability of data for its hydrodynamic coefficients (Comstock, 1977), (Bernitsas and Kekridis, 1984). A heading control law based on Nomoto's first-order model, coupled with a line of sight guidance law is chosen in order to model the fundamental behavior of closed loop conditions. Since, in practice control actions are hardly ever instantaneous, appropriate time lags are introduced in the feedback. These are modeled using the techniques outlined in (Venne, 1992). Linear eigenvalue analysis is performed in Chapter III in order to reveal parametric stability boundaries. It is established that loss of stability occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis which results in periodic solutions or limit cycles (Guckenheimer and Holmes, 1983). A nonlinear analysis based on Hopf bifurcation methods (Chow and Mallet-Paret, 1977) and (Hassard and Wan, 1978) is performed in Chapter IV. Conclusions derived from this study and some recommendations for further extensions are discussed in Chapter V. The basic programs that were used to produce the results are included in the Appendix.

## II. PROBLEM FORMULATION

### A. SHIP DYNAMICS

Restricting our attention to the horizontal plane, the mathematical model consists of the nonlinear sway (translational motion parallel to the vehicle's longitudinal axis) and yaw (rotational motion about the vertical axis ) equations of motion. If we consider a moving Cartesian coordinate frame located at the geometric center of the vehicle, Newton's equations of motion are:

$$m(\dot{v} + ur + x_G \dot{r}) = Y, \quad (1)$$

$$I_Z \dot{r} + m x_G (\dot{v} + ur) = N, \quad (2)$$

where  $v$  and  $r$  are the relative sway and yaw velocities of the moving vehicle with respect to the water,  $m$  is the vehicle's mass,  $I_Z$  is its moment of inertia with respect to the vertical axis, and  $u$  is the constant forward speed. Since all quantities will be considered as dimensionless in this study, in the following we consider  $u$  to be equal to one.  $x_G$  is the longitudinal position of the ship's center of gravity with respect to its centroid, and  $Y, N$  represent the total excitation sway force and yaw moment respectively. Following standard ship maneuvering assumptions, these forces can be expressed as the sum of quadratic drag terms and first-order memoryless polynomials in  $v$  and  $r$ , which represent added mass and damping due to water. In this way they can

be expressed by:

$$Y = Y_{\dot{v}}\dot{v} + Y_{\dot{r}}\dot{r} + Y_v v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 + \frac{1}{2}Y_{rvv}rv^2 + Y_r r + Y(\psi, y) + Y_{\delta}\delta, \quad (3)$$

$$N = N_{\dot{v}}\dot{v} + N_{\dot{r}}\dot{r} + N_v v + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 + \frac{1}{2}N_{rvv}rv^2 + N_r r + N(\psi, y) + N_{\delta}\delta, \quad (4)$$

where  $Y_a$ ,  $N_a$  represent the partial derivatives with respect to the indicated variable  $a$  and  $\delta$  is the effective rudder angle.  $Y(\psi, y)$ ,  $N(\psi, y)$  represent the interaction/proximity forces and moments that arise due to the presence of the canal, and shallow water effects.

The resulting nonlinear differential equations can be non-dimensionalized with respect to the constant forward speed of the ship,  $u$  and its length,  $l$ . The dimensionless time variable is then equal to  $tu/l$ . A standard inertial system  $(x, y)$  is introduced here where the  $x$ -axis points in the assumed nominal straight line path and the  $y$ -axis is the distance from this nominal path, see Figure 1. We assume that the nominal straight line path is along the centerline of the canal. In cases where the concept of a geometric centerline is not applicable, we assume that the nominal path is along the zero bank suction location. This is consistent with recommended navigation practices that are currently in use.

The complete model is presented by the following equations:

$$(m - Y_{\dot{v}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} = Y_v v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 +$$

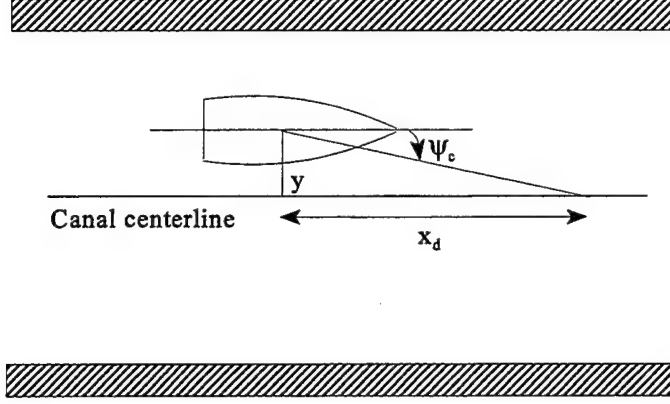


Figure 1: Vehicle geometry and definitions of symbols

$$\frac{1}{2}Y_{rvv}rv^2 + (Y_r - m)r + Y(\psi, y) + Y_\delta\delta, \quad (5)$$

$$-(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} = N_vv + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 + ,$$

$$\frac{1}{2}N_{rvv}rv^2 + (N_r - mx_G)r + N(\psi, y) + N_\delta\delta, \quad (6)$$

$$\dot{\psi} = r, \quad (7)$$

$$\dot{y} = \sin \psi + v \cos \psi, \quad (8)$$

where the expressions for the ship's yaw rate as well as its inertial position rate have been added, and  $\psi$  is the local heading angle. The interaction/proximity forces and moments are expanded to include up to third order terms,

$$Y(\psi, y) = Y_\psi\psi + Y_yy + \frac{1}{6}Y_{\psi\psi\psi}\psi^3 + \frac{1}{6}Y_{yyy}y^3, \quad (9)$$

$$N(\psi, y) = N_\psi\psi + N_yy + \frac{1}{6}N_{\psi\psi\psi}\psi^3 + \frac{1}{6}N_{yyy}y^3, \quad (10)$$

as explained in more detail in the following section.

## B. HYDRODYNAMIC COEFFICIENTS

We chose a Mariner class ship as a representative model. Its hydrodynamic coefficients and geometric properties were taken from (Comstock, 1977) and (Bernitsas and Kekridis, 1984). Results from (Beck, 1976) were used in order to model the bank suction forces and moments. Typical results are shown in Figure 2. These show force and moment coefficients versus ship deviation ( $\eta$ ), and for canal width  $w = 0.4L$ . Force and moment coefficients are nondimensionalized with respect to the water density and the ship's speed and length, as is customary in ship maneuvering. Since the suction force and moment must change their sign as either  $y$  or  $\psi$  changes its sign, they must be modeled by odd power polynomials, as was done in the previous section. Numerical values for the coefficients  $Y_y$ ,  $Y_{yyy}$ ,  $N_y$ , and  $N_{yyy}$  can be found by curve fitting of the results of Figure 2. Using a depth to draft ratio of 1.9 we were able to estimate,

$$Y_y = 0.02 ,$$

$$Y_{yyy} = 0.468 ,$$

$$N_y = -0.0025 ,$$

$$N_{yyy} = 0 .$$

The value of  $N_\psi$  was estimated by "discretizing" the ship in two segments,

FIGURE 13  
VARIATION OF SIDE FORCE AND MOMENT WITH  
WALL POSITION RATIO FOR THE MARINER

$w/L = .4$

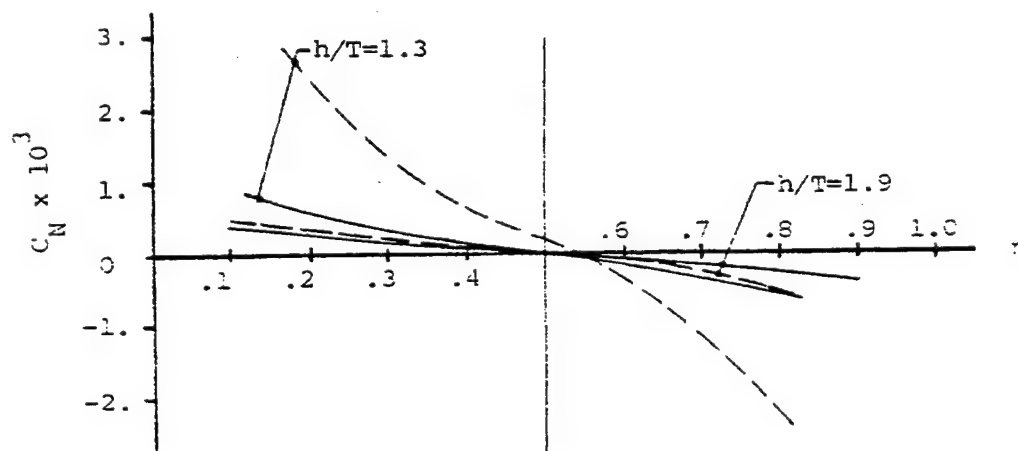
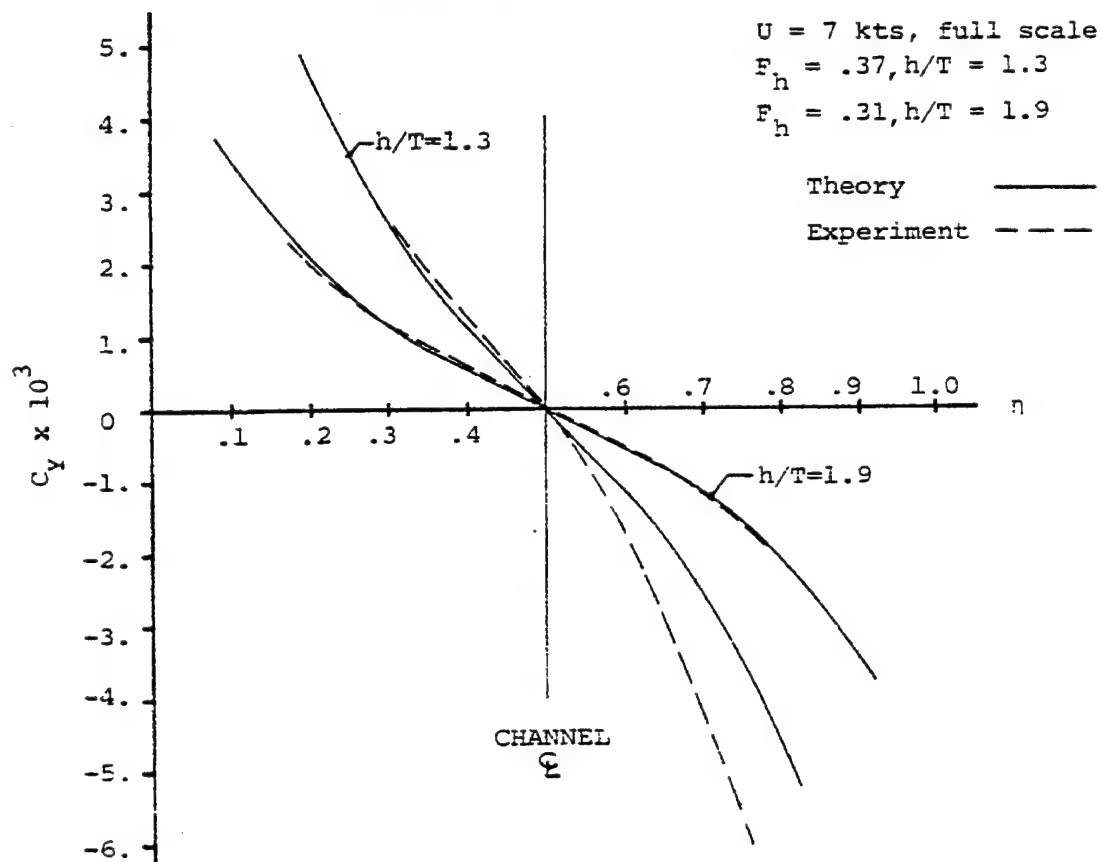


Figure 2: Forces and moments due to canal [Beck (1976)]

at the bow and the stern, and calculating the suction forces on each part individually. Their resultant moment produced,

$$N_\psi = 0.01 .$$

The value of  $N_{\psi\psi\psi}$  was taken to be zero, due to lack of reliable data. The value of  $Y_\psi$  was estimated to be equal to  $-Y_v = 0.014$  which is true for motions along the centerline of the canal (Comstock, 1977). Again, due to lack of reliable data we took  $Y_{\psi\psi\psi} = 0$ .

### C. CONTROL LAW

The linearized set of equations (5) through (7) can be expressed in the following form:

$$\begin{aligned}\dot{\psi} &= r , \\ \dot{v} &= a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta , \\ \dot{r} &= a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta ,\end{aligned}\tag{11}$$

where the coefficients  $a_{ij}$ ,  $b_i$  are functions of the vehicle hydrodynamic coefficients and geometric properties and are given below:

$$\begin{aligned}a_{11} &= \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_\psi + (Y_{\dot{r}} - mx_G)N_\psi] , \\ a_{12} &= \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_v + (Y_{\dot{r}} - mx_G)N_v] , \\ a_{13} &= \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})(Y_r - m) + (Y_{\dot{r}} - mx_G)(N_r - mx_G)] , \\ a_{14} &= \frac{1}{D_{vr}}[(I_Z - N_{\dot{r}})Y_y + (Y_{\dot{r}} - mx_G)N_y] ,\end{aligned}$$

$$\begin{aligned}
a_{21} &= \frac{1}{D_{vr}} [(N_{\dot{v}} - mx_G)Y_{\psi} + (m - Y_{\dot{v}})N_{\psi}] , \\
a_{22} &= \frac{1}{D_{vr}} [(N_{\dot{v}} - mx_G)Y_v + (m - Y_{\dot{v}})N_v] , \\
a_{23} &= \frac{1}{D_{vr}} [(N_{\dot{v}} - mx_G)(Y_r - m) + (m - Y_{\dot{v}})(N_r - mx_G)] , \\
a_{24} &= \frac{1}{D_{vr}} [(N_{\dot{v}} - mx_G)Y_y + (m - Y_{\dot{v}})N_y] , \\
b_1 &= \frac{1}{D_{vr}} [(I_Z - N_{\dot{r}})Y_{\delta} + (Y_{\dot{r}} - mx_G)N_{\delta}] , \\
b_2 &= \frac{1}{D_{vr}} [(N_{\dot{v}} - mx_G)Y_{\delta} + (m - Y_{\dot{v}})N_{\delta}] , \\
D_{vr} &= (m - Y_{\dot{v}})(I_Z - N_{\dot{r}}) - (N_{\dot{v}} - mx_G)(Y_{\dot{r}} - mx_G) .
\end{aligned}$$

A control law that could model a human operator should not be based on feedback of the side slip velocity  $v$ . Instead, it is more likely that human operators will respond to changes in the ship's heading angle  $\psi$  and rate of change of heading,  $r$ . Therefore, we choose to base our control law on Nomoto's model, which follows by assuming negligible sway velocity  $v$ ,

$$\dot{r} = ar + c\psi + b\delta , \quad (12)$$

where the coefficients are

$$a = a_{23} ,$$

$$b = b_2 ,$$

$$c = a_{21} .$$

A linear control law can be introduced in the form,

$$\delta_0 = k_1(\psi - \psi_c) + k_2r , \quad (13)$$

where  $\psi_c$  is the commanded heading angle and the gains  $k_1, k_2$  are computed such that the closed loop system (7), (12), and (13) has the desired dynamics. The existence of the difference  $(\psi - \psi_c)$  in the control law (13) has the effect of trying to point the ship's longitudinal axis towards the desired heading. The characteristic equation of the system is obtained from (7), (12), and (13) as

$$s^2 - (a + bk_2)s - (c + bk_1) = 0 .$$

If the desired characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 ,$$

the control gains can be computed from

$$\begin{aligned} k_1 &= -\frac{\omega_n^2}{b} - \frac{c}{b} , \\ k_2 &= -\frac{a + 2\zeta\omega_n}{b} . \end{aligned}$$

The natural frequency  $\omega_n$  and damping ratio  $\zeta$  are selected based on general properties of second order systems. Relatively high values of  $\omega_n$  and low values of  $\zeta$  will correspond to a responsive operator, while the opposite is true for combinations of low  $\omega_n$  and high  $\zeta$ .

Finally, in order to take into account the effect of rudder saturation, the commanded rudder angle is given by

$$\delta = \delta_{\text{sat}} \tanh \left( \frac{\delta_0}{\delta_{\text{sat}}} \right) , \quad (14)$$

where  $\delta_0$  is the slope of  $\delta$  at the origin given by (13), and  $\delta_{\text{sat}}$  is the saturation limit on  $\delta$ , typically around 0.4 radians. The hyperbolic tangent function is used instead of a hard saturation function, because of its differentiability.

#### D. GUIDANCE SCHEME

Since the previous control law stabilizes the ship to any commanded heading angle, it must be coupled with an appropriate orientation guidance scheme to provide path keeping along the desired track. The simplest guidance scheme which models some fundamental aspects of helmsman behavior is a pure pursuit guidance where the commanded heading angle equals the line of sight angle,

$$\psi_c = -\tan^{-1}\left(\frac{y}{x_d}\right), \quad (15)$$

as shown in Figure 1. The ship is located at  $(x, y)$  and attempts to point its longitudinal axis towards a target point which is located ahead of the ship on the reference path at a constant preview distance  $x_d$ . Pursuit guidance is achieved by commanding a heading angle  $\psi_c$  equal to the line of sight angle (15).

The positional error information  $y$  in equation (15), is assumed to lag the actual error  $y$  by an amount of  $T$  seconds, in other words,

$$y = y(t - T). \quad (16)$$

In this equation, the time lag  $T$  models the necessary time that it takes for the helmsman to process his path deviation and initiate appropriate corrective actions.



### III. LINEAR ANALYSIS

In this Chapter, we present a linearized analysis of the equations of motion. The purpose of this analysis is to assess stability or instability of the equations in response to small deviations from the straight line reference track. No attempt will be made here to characterize the transient response of the system. This can be accomplished through numerical simulations.

#### A. COMBINED SYSTEM

If we incorporate the interaction/proximity forces (9) and (10) in the ship dynamics model, equations (5) through (8), we end up with the combined system,

$$\begin{aligned}
 \dot{\psi} &= r, \\
 (m - Y_{\dot{\psi}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} &= Y_v v + \frac{1}{6}Y_{vvv}v^3 + \frac{1}{2}Y_{vrr}vr^2 + \frac{1}{2}Y_{rvv}rv^2 + \\
 &\quad (Y_r - m)r + Y_{\psi}\psi + Y_y y + \frac{1}{6}Y_{\psi\psi\psi}\psi^3 + \frac{1}{6}Y_{yyy}y^3 + Y_{\delta}\delta, \\
 -(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} &= N_v v + \frac{1}{6}N_{vvv}v^3 + \frac{1}{2}N_{vrr}vr^2 + \\
 &\quad \frac{1}{2}N_{rvv}rv^2 + (N_r - mx_G)r + N_{\psi}\psi + N_y y + \frac{1}{6}N_{\psi\psi\psi}\psi^3 + \\
 &\quad \frac{1}{6}N_{yyy}y^3 + N_{\delta}\delta, \\
 \dot{y} &= \sin \psi + v \cos \psi.
 \end{aligned} \tag{17}$$

Study of the asymptotic properties of this system is the subject of this and the following chapters.

## B. LINEARIZATION

The linearized form of equations (17) is the following,

$$\begin{aligned}
 \dot{\psi} &= r, \\
 (m - Y_{\dot{v}})\dot{v} - (Y_{\dot{r}} - mx_G)\dot{r} &= Y_v v + (Y_r - m)r + \\
 &\quad Y_{\psi}\psi + Y_y y + Y_{\delta}\delta, \\
 -(N_{\dot{v}} - mx_G)\dot{v} + (I_z - N_{\dot{r}})\dot{r} &= N_v v + (N_r - mx_G)r + \\
 &\quad N_{\psi}\psi + N_y y + N_{\delta}\delta, \\
 \dot{y} &= \psi + v.
 \end{aligned} \tag{18}$$

The rudder angle  $\delta$  has one of the forms that are developed below. The time delay operator can be expressed in terms of its Taylor series expansion,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} - \frac{1}{6}T^3y^{(iii)} + \dots \tag{19}$$

Practical computations can be performed by truncating equation (19) to first, second, or third order.

### 1. First order approximation in $y$

We have,

$$y(t - T) = y - T\dot{y},$$

or

$$y(t - T) = y - T(\psi + v). \tag{20}$$

In its linear form,

$$\tanh\left(\frac{\delta_0}{\delta_{\text{sat}}}\right) = \frac{\delta_0}{\delta_{\text{sat}}},$$

and, therefore,

$$\delta = \delta_0.$$

Furthermore, in its linear form equation (15) can be written as

$$\psi_c = -\frac{y(t - T)}{x_d}.$$

If we incorporate (20) into (13), we derive the linearized first order approximation in  $\delta$ ,

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) . \quad (21)$$

## 2. Second order approximation in $y$

Keeping the second order terms in  $y(t - T)$  we get,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} . \quad (22)$$

If we incorporate (22) into (13) along with the linear equations  $\delta = \delta_0$  and  $\psi_c = -y(t - T)/x_d$ , we get

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) + \frac{k_1T^2}{2x_d}(r + \dot{v}) . \quad (23)$$

## 3. Third order approximation in $y$

In this case,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} - \frac{1}{6}T^3y^{(iii)} , \quad (24)$$

and

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - \frac{k_1T}{x_d}(\psi + v) + \frac{k_1T^2}{2x_d}(r + \dot{v}) - \frac{k_1T^3}{6x_d}(\dot{r} + \ddot{v}) . \quad (25)$$

## 4. First order approximation in $\delta$

If a time lag,  $T$ , exists on  $\delta$  instead of simply  $y$ , then

$$\delta = \delta_{\text{sat}} \tanh \left( \frac{\delta_1}{\delta_{\text{sat}}} \right) ,$$

where,

$$\delta_1 = \delta_0(t - T) = \delta_0 - T\dot{\delta}_0 .$$

This equation models a time lag associated with the entire application of the necessary corrective control action and not just its positional error. Using the above equations along with the linear equations  $\delta = \delta_1$  and  $\psi_c = -y/x_d$ , we get

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - k_1Tr - \frac{k_1T}{x_d}(\psi + v) - k_2T\dot{r}. \quad (26)$$

##### 5. First order approximation in both $y$ and $\delta$

In a more general setting, we can assume that a time lag,  $T_1$ , exists in the control law  $\delta$ , and a different time lag,  $T_2$ , is present in the processing of the positional error  $y$ . Assuming a first order approximation for both, we have

$$\delta_1 = \delta_0(t - T_1) = \delta_0 - T_1\dot{\delta}_0,$$

and

$$y(t - T_2) = y - T_2\dot{y}.$$

Therefore, in this case using the above equations along with the linear equations  $\delta = \delta_1$  and  $\psi_c = -y(t - T_2)/x_d$ , we obtain

$$\begin{aligned} \delta = k_1\psi + k_2r + \frac{k_1}{x_d}y - k_1T_1r - \frac{k_1T_1}{x_d}(\psi + v) - \\ \frac{k_1T_2}{x_d}(\psi + v) + \frac{k_1T_1T_2}{x_d}(r + \dot{v}) - k_2T_1\dot{r}. \end{aligned} \quad (27)$$

Using one of the above expressions, the linearized equations of motion can be written as

$$\begin{aligned} \dot{\psi} &= r, \\ \dot{v} &= a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta, \\ \dot{r} &= a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta, \\ \dot{y} &= \psi + v, \end{aligned} \quad (28)$$

where all coefficients have been previously defined.

## C. SERIES EXPANSIONS OF TIME LAG

### 1. First order approximation in $y$

In this case we have,

$$y(t - T) = y - T\dot{y}.$$

Substitution of (21) into (28) yields the following matrix system,

$$\begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,1} & A_{22,1} & A_{23,1} & A_{24,1} \\ A_{31,1} & A_{32,1} & A_{33,1} & A_{34,1} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \quad (29)$$

where,

$$\begin{aligned} A_{21,1} &= a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d} \\ A_{22,1} &= a_{12} - \frac{b_1 k_1 T}{x_d} \\ A_{23,1} &= a_{13} + b_1 k_2 \\ A_{24,1} &= a_{14} + \frac{b_1 k_1}{x_d} \\ A_{31,1} &= a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d} \\ A_{32,1} &= a_{22} - \frac{b_2 k_1 T}{x_d} \\ A_{33,1} &= a_{23} + b_2 k_2 \\ A_{34,1} &= a_{24} + \frac{b_2 k_1}{x_d} \end{aligned}$$

### 2. Second order approximation in $y$

In this case we have,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y}.$$

Substitution of (23) into (28) yields the following matrix system,

$$\begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,2} & A_{22,2} & A_{23,2} & A_{24,2} \\ A_{31,2} & A_{32,2} & A_{33,2} & A_{34,2} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \quad (30)$$

where,

$$\begin{aligned} A_{21,2} &= \frac{a_{11}x_d + b_1k_1x_d - b_1k_1T}{x_d - 0.5b_1k_1T^2} \\ A_{22,2} &= \frac{a_{12}x_d - b_1k_1T}{x_d - 0.5b_1k_1T^2} \\ A_{23,2} &= \frac{a_{13}x_d + b_1k_2x_d + 0.5b_1k_1T^2}{x_d - 0.5b_1k_1T^2} \\ A_{24,2} &= \frac{a_{14}x_d + b_1k_1}{x_d - 0.5b_1k_1T^2} \\ A_{31,2} &= a_{21} + b_2k_1 - \frac{b_2k_1T}{x_d} + \frac{b_2k_1T^2}{2x_d}A_{21,2} \\ A_{32,2} &= a_{22} - \frac{b_2k_1T}{x_d} + \frac{b_2k_1T^2}{2x_d}A_{22,2} \\ A_{33,2} &= a_{23} + b_2k_2 + \frac{b_2k_1T^2}{2x_d} + \frac{b_2k_1T^2}{2x_d}A_{23,2} \\ A_{34,2} &= a_{24} + \frac{b_2k_1}{x_d} + \frac{b_2k_1T^2}{2x_d}A_{24,2} \end{aligned}$$

### 3. Third order approximation in $y$

A third order approximation in  $y$  is based on,

$$y(t - T) = y - T\dot{y} + \frac{1}{2}T^2\ddot{y} - \frac{1}{6}T^3y^{(iii)}.$$

Similar algebraic steps as in the previous two approximations result in the following eigenvalue problem,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & B_{33,3} & 0 & B_{35,3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & B_{53,3} & 0 & B_{55,3} \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \\ \dot{v}_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ A_{31,3} & A_{32,3} & A_{33,3} & A_{34,3} & A_{35,3} \\ 1 & 1 & 0 & 0 & 0 \\ A_{51,3} & A_{52,3} & A_{53,3} & A_{54,3} & A_{55,3} \end{bmatrix} \begin{bmatrix} \psi \\ v_1 \\ r \\ y \\ v_2 \end{bmatrix}, \quad (31)$$

where,  $v_1 = v$ ,  $v_2$  is the rate of change of  $v$ , and the entries of the generalized eigenvalue problem (31) are given below. Higher order approximations in  $T$  can not produce any usable stability results, since the B matrix in (31) becomes singular.

$$\begin{aligned} A_{31,3} &= a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d} \\ A_{32,3} &= a_{12} - \frac{b_1 k_1 T}{x_d} \\ A_{33,3} &= a_{13} + b_1 k_2 + \frac{b_1 k_1 T^2}{2x_d} \\ A_{34,3} &= a_{14} + \frac{b_1 k_1}{x_d} \\ A_{35,3} &= \frac{b_1 k_1 T^2}{2x_d} - 1 \\ A_{51,3} &= a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d} \\ A_{52,3} &= a_{22} - \frac{b_2 k_1 T}{x_d} \\ A_{53,3} &= a_{23} + b_2 k_2 + \frac{b_2 k_1 T^2}{2x_d} \\ A_{54,3} &= a_{24} + \frac{b_2 k_1}{x_d} \\ A_{55,3} &= \frac{b_2 k_1 T^2}{2x_d} \\ B_{33,3} &= \frac{b_2 k_1 T^3}{6x_d} \\ B_{35,3} &= B_{33,3} \\ B_{53,3} &= \frac{b_2 k_1 T^3}{6x_d} + 1 \end{aligned}$$

$$B_{55,3} = \frac{b_2 k_1 T^3}{6x_d}$$

#### 4. First order approximation in $\delta$

Substitution of (26) into (28) yields the following matrix system,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & B_{23,4} & 0 \\ 0 & 0 & B_{33,4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,4} & A_{22,4} & A_{23,4} & A_{24,4} \\ A_{31,4} & A_{32,4} & A_{33,4} & A_{34,4} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \quad (32)$$

where,

$$\begin{aligned} A_{21,4} &= a_{11} + b_1 k_1 - \frac{b_1 k_1 T}{x_d} \\ A_{22,4} &= a_{12} - \frac{b_1 k_1 T}{x_d} \\ A_{23,4} &= a_{13} + b_1 k_2 - b_1 k_1 T \\ A_{24,4} &= a_{14} + \frac{b_1 k_1}{x_d} \\ A_{31,4} &= a_{21} + b_2 k_1 - \frac{b_2 k_1 T}{x_d} \\ A_{32,4} &= a_{22} - \frac{b_2 k_1 T}{x_d} \\ A_{33,4} &= a_{23} + b_2 k_2 - b_2 k_1 T \\ A_{34,4} &= a_{24} + \frac{b_2 k_1}{x_d} \\ B_{23,4} &= b_1 k_2 T \\ B_{33,4} &= 1 + b_2 k_2 T \end{aligned}$$

#### 5. First order approximation in both $y$ and $\delta$

Substitution of (27) into (28) yields the following matrix system,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & B_{22,5} & B_{23,5} & 0 \\ 0 & B_{32,5} & B_{33,5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{v} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ A_{21,5} & A_{22,5} & A_{23,5} & A_{24,5} \\ A_{31,5} & A_{32,5} & A_{33,5} & A_{34,5} \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ v \\ r \\ y \end{bmatrix}, \quad (33)$$

where,

$$\begin{aligned}
A_{21,5} &= a_{11} + b_1 k_1 - \frac{b_1 k_1 T_1}{x_d} - \frac{b_1 k_1 T_2}{x_d} \\
A_{22,5} &= a_{12} - \frac{b_1 k_1 T_1}{x_d} - \frac{b_1 k_1 T_2}{x_d} \\
A_{23,5} &= a_{13} + b_1 k_2 - b_1 k_1 T_1 + \frac{b_1 k_1 T_1 T_2}{x_d} \\
A_{24,5} &= a_{14} + \frac{b_1 k_1}{x_d} \\
A_{31,5} &= a_{21} + b_2 k_1 - \frac{b_2 k_1 T_1}{x_d} - \frac{b_2 k_1 T_2}{x_d} \\
A_{32,5} &= a_{22} - \frac{b_2 k_1 T_1}{x_d} - \frac{b_2 k_1 T_2}{x_d} \\
A_{33,5} &= a_{23} + b_2 k_2 - b_2 k_1 T_1 + \frac{b_2 k_1 T_1 T_2}{x_d} \\
A_{34,5} &= a_{24} + \frac{b_2 k_1}{x_d} \\
B_{22,5} &= 1 - \frac{b_1 k_1 T_1 T_2}{x_d} \\
B_{23,5} &= b_1 k_2 T_1 \\
B_{32,5} &= -\frac{b_2 k_1 T_1 T_2}{x_d} \\
B_{33,5} &= 1 + b_2 k_2 T_1
\end{aligned}$$

#### D. EXACT COMPUTATION OF TIME LAG

The previous analysis using Taylor series expansions of  $y(t - T)$  breaks down for approximations beyond third order, as the corresponding generalized eigenvalue problem becomes singular. Thus, in order to obtain an exact computation of the stability curves and check the validity of the calculations, a different technique will be performed in this section. This technique is based

on frequency response methods and it utilizes Nyquist's criterion for stability [Friedland (1986)]. We write the system of equations (11) along with equations  $\dot{y} = \psi + v$  and  $\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y(t - T)$ , in the Laplace domain, where

$$y(t - T) \longrightarrow ye^{-Ts}, \quad (34)$$

is the time delay operator. The characteristic equation of the system is,

$$s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4 + (\beta_2 s^2 + \beta_1 s + \beta_0) \frac{1}{x_d} e^{-Ts} = 0, \quad (35)$$

where

$$\begin{aligned} \alpha_1 &= -a_{12} - a_{23} - b_2 k_2, \\ \alpha_2 &= -a_{14} + a_{12} a_{23} - a_{22} b_1 k_2 - a_{22} a_{13} + a_{12} b_2 k_2 - b_2 k_1 - a_{21}, \\ \alpha_3 &= -a_{24} - a_{13} a_{24} - a_{24} b_1 k_2 + a_{23} a_{14} + a_{14} b_2 k_2 - b_1 k_1 a_{22} - \\ &\quad a_{11} a_{22} + b_2 k_1 a_{12} + a_{21} a_{12}, \\ \alpha_4 &= a_{21} a_{14} + a_{12} a_{24} - a_{22} a_{14} - a_{11} a_{24} + b_2 k_1 a_{14} - b_1 k_1 a_{24}, \\ \beta_2 &= -b_1 k_1, \\ \beta_1 &= -a_{13} b_2 k_1 + b_1 k_1 a_{23} - b_2 k_1, \\ \beta_0 &= a_{12} b_2 k_1 - b_1 k_1 a_{22} - b_2 k_1 a_{11} + b_1 k_1 a_{21}, \end{aligned}$$

The characteristic equation is written as,

$$1 + \frac{1}{x_d} G(s) = 0,$$

where

$$G(s) = \frac{(\beta_2 s^2 + \beta_1 s + \beta_0) e^{-Ts}}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \quad (36)$$

is the open loop transfer function , and  $\frac{1}{x_d}$  denotes the effective position gain. For stability, we can utilize the Nyquist criterion which states that the critical value of  $x_d$  can be computed from,

$$\frac{1}{x_d} |G(j\omega)| = 1 \quad \text{for} \quad \arg G(j\omega) = -\pi \quad (37)$$

where  $j$  denotes the imaginary unit. The argument,  $\arg G(j\omega)$ , of (36) is,

$$\arg G(j\omega) = -\omega T + \tan^{-1} \frac{\beta_1 \omega}{\beta_0 - \beta_2 \omega^2} - \tan^{-1} \frac{\alpha_3 \omega - \alpha_1 \omega^3}{\omega^4 - \alpha_2 \omega^2 + \alpha_4} \quad (38)$$

The set of equations (37) result in the critical visibility distance,

$$x_d = \sqrt{\frac{\beta_1^2 \omega^2 + (\beta_0 - \beta_2 \omega^2)^2}{(\alpha_3 \omega - \alpha_1 \omega^3)^2 + (\omega^4 - \alpha_2 \omega^2 + \alpha_4)^2}} \quad (39)$$

The value of  $\omega$  in (38) such that  $\arg G(j\omega) = -\pi$  is the value of the phase crossover frequency. After solving for the phase crossover frequency, the critical value of  $x_d$  is obtained by direct substitution of this value of  $\omega$  into equation (39).

## E. RESULTS AND DISCUSSION

Typical results are presented in Figures 3 through 6. All results shown are nondimensional unless otherwise stated. The critical value of  $x_d$  versus  $\omega_n$  for zero time lag, and parametrized for different values of the damping ratio  $\zeta$  is shown in Figure 3. Stability is ensured for values of  $x_d$  greater than its critical value. It can be seen that lower values of  $\omega_n$  require higher values of  $x_d$  for stability. This means that a less responsive helmsman will need a

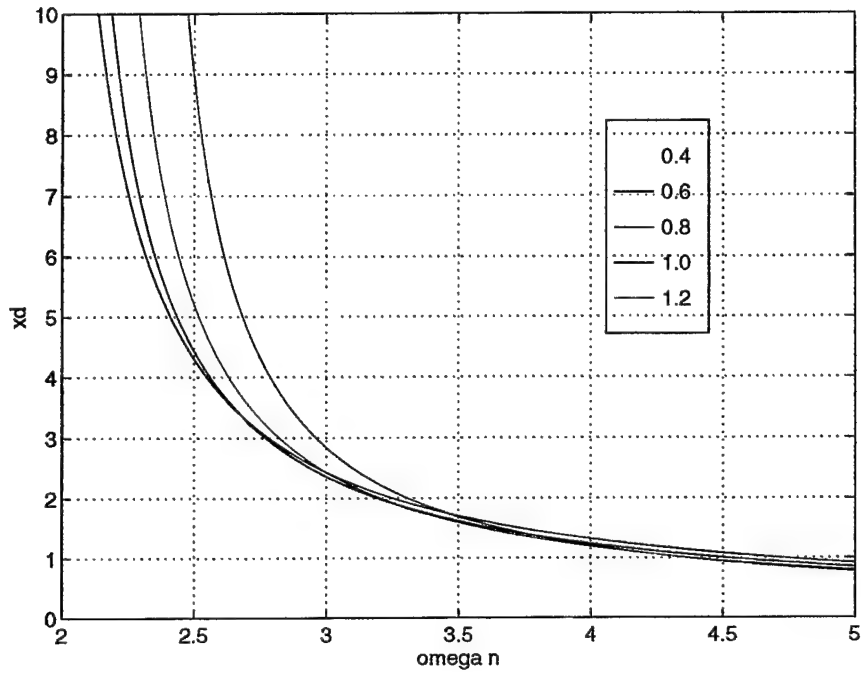


Figure 3: Critical  $x_d$  versus  $\omega_n$  for  $T_2 = 0$  and various values of  $\zeta$

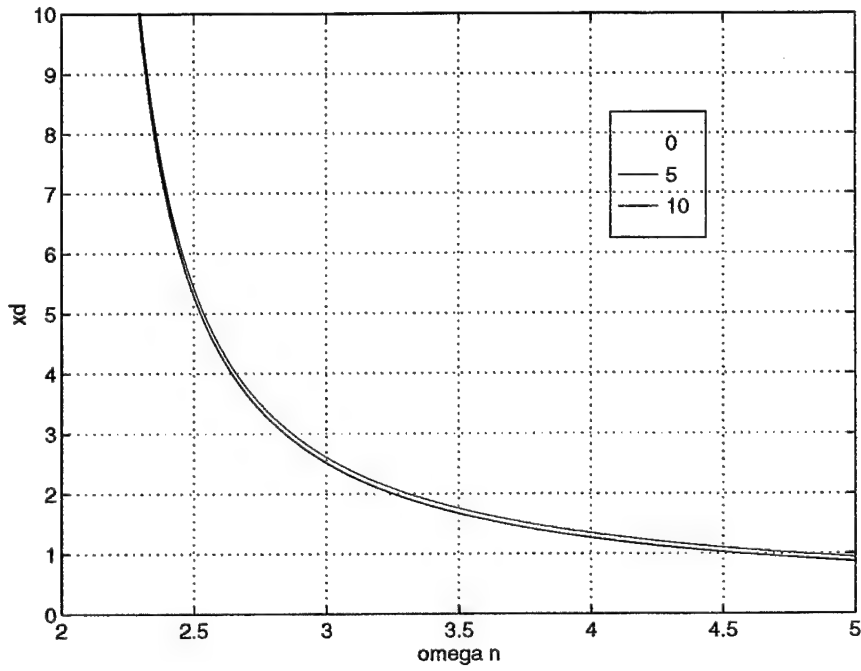


Figure 4: Critical  $x_d$  versus  $\omega_n$  for  $\zeta = 0.8$  and various values of  $T_2$

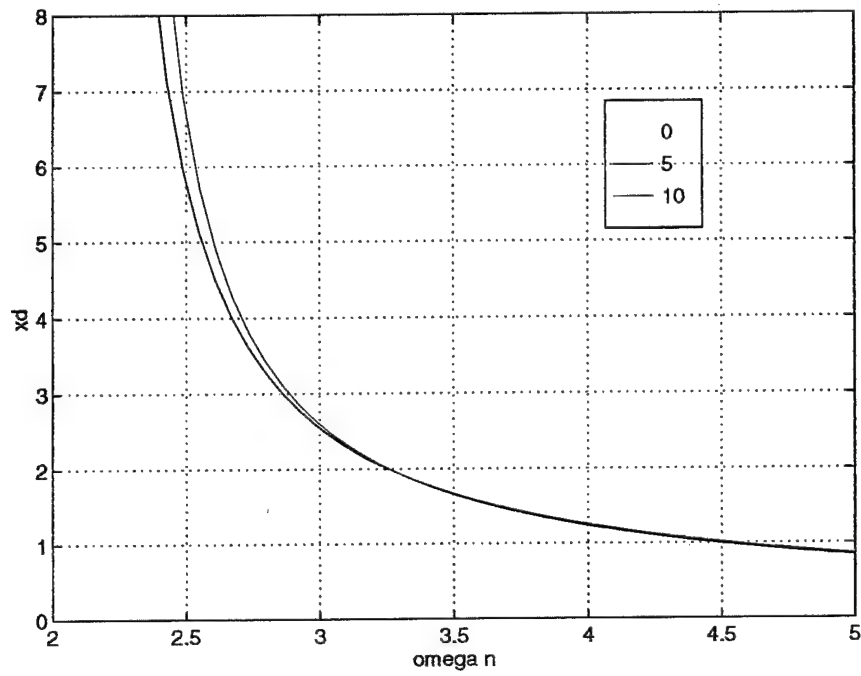


Figure 5: Critical  $x_d$  versus  $\omega_n$  for  $\zeta = 0.8$ ,  $T_2 = 5\text{sec}$  and various values of  $T_1$

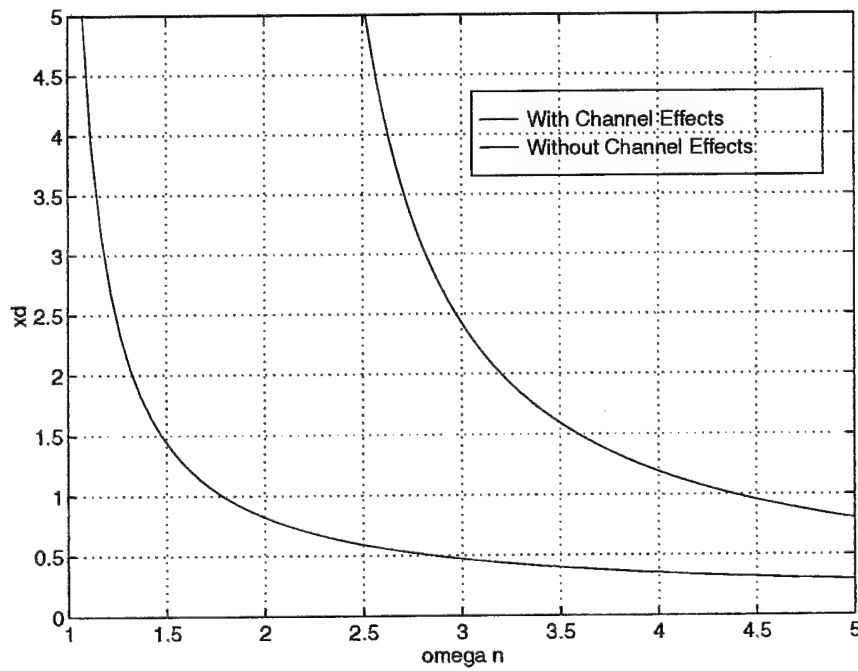


Figure 6: Critical  $x_d$  versus  $\omega_n$  for  $\zeta = 0.8$  and zero time lag: Channel effects

longer lookahead distance for a stable operation. Similar conclusions hold for variations in the damping ratio  $\zeta$ . In this case, higher values of  $\zeta$  correspond to better helmsman response which requires less lookahead distance.

The effect of time lag  $T_2$  is shown in Figure 4. Time lags are in seconds. It can be seen that reasonable amounts of time lag do not have a serious effect on stability, at least in a linear sense. Of course, an amount of time lag may have a serious effect on the transient response of the system as well as its ability to reject an external disturbance. This can only be established by a systematic series of numerical simulations. Similar conclusions hold for non-zero time lags  $T_1$ , as shown in Figure 5.

Finally, the severe destabilizing effect of the canal is demonstrated by the results of Figure 6 where a comparison between canal and open water results is presented. It can be seen that an order of magnitude increase in the lookahead distance may be required if the same control parameters are to be used in both open water and a canal.

## IV. NONLINEAR ANALYSIS

### A. INTRODUCTION

It can be numerically confirmed that in all cases of stability loss of the previous chapter, one pair of complex conjugate eigenvalues of the corresponding eigenvalue problem crosses transversally the imaginary axis. A situation like this in which a certain parameter is varied such that the real part of one pair of complex conjugate eigenvalues of the linearized system matrix crosses zero, results in the system leaving its steady state in an oscillatory manner. This loss of stability is called Hopf bifurcation and generically occurs in one of two ways, supercritical or subcritical. In the supercritical case, stable limit cycles are generated after the nominal straight line motion loses its stability. The amplitudes of these limit cycles are continuously increasing as the parameter distance from its critical value is increased. For small values of this criticality distance the resulting limit cycle is of small amplitude and differs little from the initial nominal state. In the subcritical case, however, stable limit cycles are generated before the nominal state loses its stability. Therefore, depending on the initial conditions it is possible to diverge away from the nominal straight line path and converge towards a limit cycle even before the nominal motion loses its stability. This means that in the subcritical Hopf bifurcation case the domain of attraction of the nominal state is decreasing and in fact it shrinks to zero as the critical point is approached. Random external dis-

turbances of sufficient magnitude can throw the vehicle off to an oscillatory steady state even though the nominal state may still remain stable. After the nominal state becomes unstable, a discontinuous increase in the magnitude of motions is observed as there exist no simple stable nearby attractors for the vehicle to converge to. Distinction between these two qualitatively different types of bifurcation is, therefore, essential in design. The computational procedure requires higher order approximations in the equations of motion and is the subject of the next section.

## B. DETAILED CALCULATIONS

The nonlinear equations of motion are written as,

$$\dot{\psi} = r, \quad (40)$$

$$\dot{v} = a_{11}\psi + a_{12}v + a_{13}r + a_{14}y + b_1\delta', \quad (41)$$

$$\dot{r} = a_{21}\psi + a_{22}v + a_{23}r + a_{24}y + b_2\delta', \quad (42)$$

$$\dot{y} = \sin \psi + v \cos \psi, \quad (43)$$

where the control law is,

$$\delta = k_1\psi + k_2r + k_1 \tan^{-1} \frac{y(t-T)}{x_d}, \quad (44)$$

and, including saturation,

$$\delta' = \delta_{\text{sat}} \tanh \left( \frac{\delta}{\delta_{\text{sat}}} \right). \quad (45)$$

We perform a Taylor series expansion of the equations, keeping the first non-vanishing nonlinear terms. Due to port/starboard symmetry in the problem

this means expansion to third order terms,

$$\sin \psi = \psi - \frac{1}{6}\psi^3, \quad \cos \psi = 1 - \frac{1}{2}\psi^2, \quad (46)$$

$$\delta' = \delta - \frac{1}{3\delta_{\text{sat}}^2}\delta^3, \quad (47)$$

$$\delta = k_1\psi + k_2r + \frac{k_1}{x_d}y(t-T) - \frac{k_1}{3x_d^3}y^3(t-T), \quad (48)$$

where for consistency, the term  $y(t-T)$  is approximated by its first order expansion in  $T$ ,

$$y(t-T) = y - T\dot{y} = y - T\psi + \frac{1}{6}T\psi^3 - Tv + \frac{1}{2}Tv\psi^2. \quad (49)$$

Substitution of equations (46) to (49) into equations (40) to (45), produces the system,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}), \quad (50)$$

where the state variables vector is,

$$\mathbf{x} = [\psi, v, r, y]^T,$$

$\mathbf{A}$  is the linearized matrix at equilibrium, and  $\mathbf{g}(\mathbf{x})$  contains all third order terms.

If  $\mathbf{T}$  is the matrix of eigenvectors of  $\mathbf{A}$  evaluated at the critical point  $x_d$ , the linear change of coordinates,

$$\mathbf{x} = \mathbf{T}\mathbf{z}, \quad \mathbf{z} = \mathbf{T}^{-1}\mathbf{x}, \quad (51)$$

transforms system (50) into its normal coordinate form,

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{z} + \mathbf{T}^{-1}\mathbf{g}(\mathbf{T}\mathbf{z}). \quad (52)$$

At the Hopf bifurcation point, matrix  $\mathbf{T}^{-1}\mathbf{AT}$  takes the form,

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & -\omega_0 & 0 & 0 \\ \omega_0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & q \end{bmatrix},$$

where  $\omega_0$  is the imaginary part of the critical pair of eigenvalues, and the remaining two eigenvalues  $p$  and  $q$  are negative (or complex conjugate with negative real parts). Therefore in the new system of coordinates (51) the dynamics of (50) are governed by a reduced two-dimensional system  $z_1$  and  $z_2$  since the coordinates  $z_3$  and  $z_4$  correspond to the eigenvalues  $p$  and  $q$  and are asymptotically stable. For values of  $x_d$  close to the bifurcation point  $x_d$ , matrix  $\mathbf{T}^{-1}\mathbf{AT}$  is,

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} \alpha'\varepsilon & -(\omega_0 + \omega'\varepsilon) & 0 & 0 \\ \omega_0 + \omega'\varepsilon & \alpha'\varepsilon & 0 & 0 \\ 0 & 0 & p + p'\varepsilon & 0 \\ 0 & 0 & 0 & q + q'\varepsilon \end{bmatrix},$$

where,  $\varepsilon$  denotes the criticality difference,

$$\varepsilon = x_d - x_{d_{\text{critical}}}, \quad (53)$$

$\omega'$  = derivative of the real part of the critical eigenvalues with respect to  $\varepsilon$ ,

$\alpha'$  = derivative of the imaginary part of the critical eigenvalues

with respect to  $\varepsilon$ ,

$p'$  = derivative of  $p$  with respect to  $\varepsilon$ ,

$q'$  = derivative of  $q$  with respect to  $\varepsilon$ .

Due to continuity, the eigenvalues  $p + p'\varepsilon$  and  $q + q'\varepsilon$  remain negative for small nonzero values of  $\varepsilon$ . Therefore, the coordinates  $z_3, z_4$  correspond to negative

eigenvalues and are asymptotically stable. Center manifold theory predicts that the relationship between the critical coordinates  $z_1, z_2$  and the stable coordinates  $z_3, z_4$  is at least of quadratic order. In fact, due to the symmetry in our problem the relationship is cubic,

$$z_1 = \mathcal{O}(z_3^3, z_4^3), \quad z_2 = \mathcal{O}(z_3^3, z_4^3).$$

It follows that because of this order,  $z_3, z_4$  do not influence the third order Taylor series expansions, and can be dropped from the equations. Therefore, we can write the reduced system that describes the essential dynamics of (52) in the form,

$$\dot{z}_1 = \alpha' \varepsilon z_1 - (\omega_0 + \omega' \varepsilon) z_2 + F_1(z_1, z_2), \quad (54)$$

$$\dot{z}_2 = (\omega_0 + \omega' \varepsilon) z_1 + \alpha' \varepsilon z_2 + F_2(z_1, z_2), \quad (55)$$

where  $F_1, F_2$  contain the third order terms,

$$F_1(z_1, z_2) = r_{11} z_1^3 + r_{12} z_1^2 z_2 + r_{13} z_1 z_2^2 + r_{14} z_2^3, \quad (56)$$

$$F_2(z_1, z_2) = r_{21} z_1^3 + r_{22} z_1^2 z_2 + r_{23} z_1 z_2^2 + r_{24} z_2^3. \quad (57)$$

The coefficients  $r_{ij}$  are computable from the previous Taylor expansions, and they are given below.

The nonlinear expansion coefficients  $r_{ij}$  that are utilized in the definition of the cubic stability coefficient  $\mathcal{K}$  are given by the following:

$$r_{11} = n_{12} \ell_{21} + n_{13} \ell_{31} + n_{14} \ell_{41} + n_{12} d_{11} + n_{13} d_{21},$$

$$r_{12} = n_{12} \ell_{22} + n_{13} \ell_{32} + n_{14} \ell_{42} + n_{12} d_{12} + n_{13} d_{22},$$

$$\begin{aligned}
r_{13} &= n_{12}\ell_{23} + n_{13}\ell_{33} + n_{14}\ell_{43} + n_{12}d_{13} + n_{13}d_{23} , \\
r_{14} &= n_{12}\ell_{24} + n_{13}\ell_{34} + n_{14}\ell_{44} + n_{12}d_{14} + n_{13}d_{24} , \\
r_{21} &= n_{22}\ell_{21} + n_{23}\ell_{31} + n_{24}\ell_{41} + n_{22}d_{11} + n_{23}d_{21} , \\
r_{22} &= n_{22}\ell_{22} + n_{23}\ell_{32} + n_{24}\ell_{42} + n_{22}d_{12} + n_{23}d_{22} , \\
r_{23} &= n_{22}\ell_{23} + n_{23}\ell_{33} + n_{24}\ell_{43} + n_{22}d_{13} + n_{23}d_{23} , \\
r_{24} &= n_{22}\ell_{24} + n_{23}\ell_{34} + n_{24}\ell_{44} + n_{22}d_{14} + n_{23}d_{24} ,
\end{aligned}$$

where we denote,

$$\mathbf{T} = [m_{ij}] , \quad \mathbf{T}^{-1} = [n_{ij}] , \quad i, j = 1, \dots, 4 .$$

For numerical purposes the critical eigenvector of  $\mathbf{T}$  must be normalized so that its first nonvanishing coefficient is identically equal to 1. The coefficients  $\ell_{ij}$  are given by,

$$\begin{aligned}
\frac{\ell_{21}}{b_1} &= \delta_{\psi\psi v} m_{11}^2 m_{21} + \delta_{\psi vv} m_{11} m_{21}^2 + \delta_{\psi\psi r} m_{11}^2 m_{31} + \delta_{\psi rr} m_{11} m_{31}^2 \\
&\quad + \delta_{\psi\psi y} m_{11}^2 m_{41} + \delta_{\psi yy} m_{11} m_{41}^2 + \delta_{vvr} m_{21}^2 m_{31} + \delta_{vrr} m_{21} m_{31}^2 \\
&\quad + \delta_{vvy} m_{21}^2 m_{41} + \delta_{vyy} m_{21} m_{41}^2 + \delta_{rry} m_{31}^2 m_{41} + \delta_{ryy} m_{31} m_{41}^2 \\
&\quad + \delta_{\psi vr} m_{11} m_{21} m_{31} + \delta_{\psi vy} m_{11} m_{21} m_{41} + \delta_{\psi ry} m_{11} m_{41} m_{31} \\
&\quad + \delta_{vry} m_{21} m_{41} m_{31} + \delta_{\psi\psi\psi} m_{11}^3 + \delta_{vvv} m_{21}^3 + \delta_{rrr} m_{31}^3 + \delta_{yyy} m_{41}^3 , \\
\frac{\ell_{22}}{b_1} &= \delta_{\psi\psi v} (m_{11}^2 m_{22} + 2m_{11} m_{12} m_{21}) + \delta_{\psi vv} (m_{12} m_{21}^2 + 2m_{11} m_{12} m_{22}) \\
&\quad + \delta_{\psi\psi r} (m_{11}^2 m_{32} + 2m_{11} m_{12} m_{31}) + \delta_{\psi rr} (m_{12} m_{31}^2 + 2m_{11} m_{31} m_{32}) \\
&\quad + \delta_{\psi\psi y} (m_{11}^2 m_{42} + 2m_{11} m_{12} m_{41}) + \delta_{\psi yy} (m_{41}^2 m_{12} + 2m_{11} m_{41} m_{42}) \\
&\quad + \delta_{vvr} (m_{21}^2 m_{32} + 2m_{31} m_{21} m_{22}) + \delta_{vrr} (m_{22} m_{31}^2 + 2m_{31} m_{32} m_{21})
\end{aligned}$$

$$\begin{aligned}
& +\delta_{vvy}(m_{21}^2 m_{42} + 2m_{41} m_{21} m_{22}) + \delta_{vyy}(m_{22} m_{41}^2 + 2m_{41} m_{42} m_{21}) \\
& +\delta_{rry}(m_{31}^2 m_{42} + 2m_{41} m_{31} m_{32}) + \delta_{ryy}(m_{32} m_{41}^2 + 2m_{41} m_{42} m_{31}) \\
& +\delta_{\psi vr}[m_{11} m_{21} m_{32} + m_{31}(m_{11} m_{22} + m_{12} m_{21})] \\
& +\delta_{\psi vy}[m_{11} m_{21} m_{42} + m_{41}(m_{11} m_{22} + m_{12} m_{21})] \\
& +\delta_{\psi ry}[m_{11} m_{41} m_{32} + m_{31}(m_{11} m_{42} + m_{12} m_{41})] \\
& +\delta_{vry}[m_{21} m_{41} m_{32} + m_{31}(m_{21} m_{42} + m_{22} m_{41})] \\
& +\delta_{\psi\psi\psi}(3m_{11}^2 m_{12}) + \delta_{vvv}(3m_{21}^2 m_{22}) + \delta_{rrr}(3m_{31}^2 m_{32}) + \delta_{yyy}(3m_{41}^2 m_{42}) , \\
\frac{\ell_{23}}{b_1} = & \delta_{\psi\psi v}(m_{12}^2 m_{21} + 2m_{11} m_{12} m_{22}) + \delta_{\psi vv}(m_{11} m_{22}^2 + 2m_{12} m_{21} m_{22}) \\
& +\delta_{\psi\psi r}(m_{12}^2 m_{31} + 2m_{11} m_{12} m_{32}) + \delta_{\psi rr}(m_{11} m_{32}^2 + 2m_{12} m_{31} m_{32}) \\
& +\delta_{\psi\psi y}(m_{12}^2 m_{41} + 2m_{11} m_{12} m_{42}) + \delta_{\psi yy}(m_{42}^2 m_{11} + 2m_{12} m_{41} m_{42}) \\
& +\delta_{vvr}(m_{22}^2 m_{31} + 2m_{21} m_{22} m_{32}) + \delta_{vrr}(m_{21} m_{32}^2 + 2m_{31} m_{22} m_{32}) \\
& +\delta_{vvy}(m_{22}^2 m_{41} + 2m_{21} m_{22} m_{42}) + \delta_{vyy}(m_{21} m_{42}^2 + 2m_{41} m_{42} m_{22}) \\
& +\delta_{rry}(m_{32}^2 m_{41} + 2m_{42} m_{31} m_{32}) + \delta_{ryy}(m_{31} m_{42}^2 + 2m_{41} m_{42} m_{32}) \\
& +\delta_{\psi vr}[m_{12} m_{22} m_{31} + m_{32}(m_{11} m_{22} + m_{12} m_{21})] \\
& +\delta_{\psi vy}[m_{12} m_{22} m_{41} + m_{42}(m_{11} m_{22} + m_{12} m_{21})] \\
& +\delta_{\psi ry}[m_{12} m_{42} m_{31} + m_{32}(m_{11} m_{42} + m_{12} m_{41})] \\
& +\delta_{vry}[m_{22} m_{42} m_{31} + m_{32}(m_{21} m_{42} + m_{22} m_{41})] \\
& +\delta_{\psi\psi\psi}(3m_{11} m_{12}^2) + \delta_{vvv}(3m_{21} m_{22}^2) + \delta_{rrr}(3m_{31} m_{32}^2) + \delta_{yyy}(3m_{41} m_{42}^2) , \\
\frac{\ell_{24}}{b_1} = & \delta_{\psi\psi v} m_{12}^2 m_{22} + \delta_{\psi vv} m_{12} m_{22}^2 + \delta_{\psi\psi r} m_{12}^2 m_{32} + \delta_{\psi rr} m_{12} m_{32}^2 \\
& +\delta_{\psi\psi y} m_{12}^2 m_{42} + \delta_{\psi yy} m_{12} m_{42}^2 + \delta_{vvr} m_{22}^2 m_{32} + \delta_{vrr} m_{22} m_{32}^2 \\
& +\delta_{vvy} m_{22}^2 m_{42} + \delta_{vyy} m_{22} m_{42}^2 + \delta_{rry} m_{32}^2 m_{42} + \delta_{ryy} m_{32} m_{42}^2
\end{aligned}$$

$$\begin{aligned}
& +\delta_{\psi vr}m_{12}m_{22}m_{32} + \delta_{\psi vy}m_{12}m_{22}m_{42} + \delta_{\psi ry}m_{12}m_{42}m_{32} + \delta_{vry}m_{22}m_{42}m_{32} \\
& +\delta_{\psi\psi\psi}m_{12}^3 + \delta_{vvv}m_{22}^3 + \delta_{rrr}m_{32}^3 + \delta_{yyy}m_{42}^3 , \\
\frac{\ell_{31}}{b_2} &= \frac{\ell_{21}}{b_1} , \\
\frac{\ell_{32}}{b_2} &= \frac{\ell_{22}}{b_1} , \\
\frac{\ell_{33}}{b_2} &= \frac{\ell_{23}}{b_1} , \\
\frac{\ell_{34}}{b_2} &= \frac{\ell_{24}}{b_1} , \\
\ell_{41} &= -\frac{1}{2}m_{11}^2 \left( m_{21} + \frac{1}{3}m_{11} \right) , \\
\ell_{42} &= -m_{11} \left( m_{12}m_{21} + \frac{1}{2}m_{11}m_{22} + \frac{1}{2}m_{12}m_{11} \right) , \\
\ell_{43} &= -m_{12} \left( m_{11}m_{22} + \frac{1}{2}m_{12}m_{21} + \frac{1}{2}m_{12}m_{11} \right) , \\
\ell_{44} &= -\frac{1}{2}m_{12}^2 \left( m_{22} + \frac{1}{3}m_{12} \right) .
\end{aligned}$$

The coefficients  $d_{ii}$  are given by,

$$\begin{aligned}
d_{11} &= \frac{1}{D_{vr}}[c_{11}(I_z - N_{\dot{r}}) + c_{21}(Y_{\dot{r}} - mx_G)] , \\
d_{12} &= \frac{1}{D_{vr}}[c_{12}(I_z - N_{\dot{r}}) + c_{22}(Y_{\dot{r}} - mx_G)] , \\
d_{13} &= \frac{1}{D_{vr}}[c_{13}(I_z - N_{\dot{r}}) + c_{23}(Y_{\dot{r}} - mx_G)] , \\
d_{14} &= \frac{1}{D_{vr}}[c_{14}(I_z - N_{\dot{r}}) + c_{24}(Y_{\dot{r}} - mx_G)] , \\
d_{21} &= \frac{1}{D_{vr}}[c_{11}(N_{\dot{v}} - mx_G) + c_{21}(m - Y_{\dot{v}})] , \\
d_{22} &= \frac{1}{D_{vr}}[c_{12}(N_{\dot{v}} - mx_G) + c_{22}(m - Y_{\dot{v}})] , \\
d_{23} &= \frac{1}{D_{vr}}[c_{13}(N_{\dot{v}} - mx_G) + c_{23}(m - Y_{\dot{v}})] , \\
d_{24} &= \frac{1}{D_{vr}}[c_{14}(N_{\dot{v}} - mx_G) + c_{24}(m - Y_{\dot{v}})] ,
\end{aligned}$$

where

$$\begin{aligned}
c_{11} &= \frac{1}{6}Y_{vvv}m_{21}^3 + \frac{1}{2}Y_{vrr}m_{31}^2m_{21} + \frac{1}{2}Y_{rvv}m_{21}^2m_{31} \\
&+ \frac{1}{6}Y_{\psi\psi\psi}m_{11}^3 + \frac{1}{6}Y_{yyy}m_{41}^3, \\
c_{12} &= \frac{1}{6}Y_{vvv}3m_{21}^2m_{22} + \frac{1}{2}Y_{vrr}(m_{31}^2m_{22} + 2m_{31}m_{32}m_{21}) \\
&+ \frac{1}{2}Y_{rvv}(m_{21}^2m_{32} + 2m_{31}m_{21}m_{22}) + \frac{1}{6}Y_{\psi\psi\psi}3m_{11}^2m_{12} \\
&+ \frac{1}{6}Y_{yyy}3m_{41}^2m_{42}, \\
c_{13} &= \frac{1}{6}Y_{vvv}3m_{22}^2m_{21} + \frac{1}{2}Y_{vrr}(m_{32}^2m_{21} + 2m_{31}m_{32}m_{22}) \\
&+ \frac{1}{2}Y_{rvv}(m_{22}^2m_{31} + 2m_{32}m_{21}m_{22}) + \frac{1}{6}Y_{\psi\psi\psi}3m_{12}^2m_{11} \\
&+ \frac{1}{6}Y_{yyy}3m_{42}^2m_{41}, \\
c_{14} &= \frac{1}{6}Y_{vvv}m_{22}^3 + \frac{1}{2}Y_{vrr}m_{32}^2m_{22} + \frac{1}{2}Y_{rvv}m_{22}^2m_{32} \\
&+ \frac{1}{6}Y_{\psi\psi\psi}m_{12}^3 + \frac{1}{6}Y_{yyy}m_{42}^3, \\
c_{21} &= \frac{1}{6}N_{vvv}m_{21}^3 + \frac{1}{2}N_{vrr}m_{31}^2m_{21} + \frac{1}{2}N_{rvv}m_{21}^2m_{31} \\
&+ \frac{1}{6}N_{\psi\psi\psi}m_{11}^3 + \frac{1}{6}N_{yyy}m_{41}^3, \\
c_{22} &= \frac{1}{6}N_{vvv}3m_{21}^2m_{22} + \frac{1}{2}N_{vrr}(m_{31}^2m_{22} + 2m_{31}m_{32}m_{21}) \\
&+ \frac{1}{2}N_{rvv}(m_{21}^2m_{32} + 2m_{31}m_{21}m_{22}) + \frac{1}{6}N_{\psi\psi\psi}3m_{11}^2m_{12} \\
&+ \frac{1}{6}N_{yyy}3m_{41}^2m_{42}, \\
c_{23} &= \frac{1}{6}N_{vvv}3m_{22}^2m_{21} + \frac{1}{2}N_{vrr}(m_{32}^2m_{21} + 2m_{31}m_{32}m_{22}) \\
&+ \frac{1}{2}N_{rvv}(m_{22}^2m_{31} + 2m_{32}m_{21}m_{22}) + \frac{1}{6}N_{\psi\psi\psi}3m_{12}^2m_{11} \\
&+ \frac{1}{6}N_{yyy}3m_{42}^2m_{41}, \\
c_{24} &= \frac{1}{6}N_{vvv}m_{22}^3 + \frac{1}{2}N_{vrr}m_{32}^2m_{22} + \frac{1}{2}N_{rvv}m_{22}^2m_{32} \\
&+ \frac{1}{6}N_{\psi\psi\psi}m_{12}^3 + \frac{1}{6}N_{yyy}m_{42}^3.
\end{aligned}$$

The coefficients in a third order expansion of the control law are defined by

$$\begin{aligned}
\delta_{\psi\psi v} &= -\frac{1}{\delta_{\text{sat}}^2}(k'_1)^2 k'_2 + 0.5k_1 \frac{T}{x_d} + \frac{k_1 T^3}{x_d^3}, \\
\delta_{\psi vv} &= -\frac{1}{\delta_{\text{sat}}^2} k'_1 (k'_2)^2 + \frac{T^3 k_1}{x_d^3}, \\
\delta_{\psi\psi r} &= -\frac{1}{\delta_{\text{sat}}^2} (k'_1)^2 k_2, \\
\delta_{\psi rr} &= -\frac{1}{\delta_{\text{sat}}^2} k'_1 k_2^2, \\
\delta_{\psi\psi y} &= -\frac{1}{\delta_{\text{sat}}^2} (k'_1)^2 \frac{k_1}{x_d} - \frac{T^2 k_1}{x_d^3}, \\
\delta_{\psi yy} &= -\frac{1}{\delta_{\text{sat}}^2} k'_1 \frac{k_1^2}{x_d^2} + \frac{T k_1}{x_d^3}, \\
\delta_{vvr} &= -\frac{1}{\delta_{\text{sat}}^2} (k'_2)^2 k_2, \\
\delta_{vrr} &= -\frac{1}{\delta_{\text{sat}}^2} k'_2 k_2^2, \\
\delta_{vvy} &= -\frac{1}{\delta_{\text{sat}}^2} (k'_2)^2 \frac{k_1}{x_d} - \frac{k_1 T^2}{x_d^3}, \\
\delta_{vyy} &= -\frac{1}{\delta_{\text{sat}}^2} k'_2 \frac{k_1^2}{x_d^2} + \frac{T k_1}{x_d^3}, \\
\delta_{rry} &= -\frac{1}{\delta_{\text{sat}}^2} k_2^2 \frac{k_1}{x_d}, \\
\delta_{ryy} &= -\frac{1}{\delta_{\text{sat}}^2} k_2 \frac{k_1^2}{x_d^2}, \\
\delta_{\psi vr} &= -\frac{1}{\delta_{\text{sat}}^2} 2k'_1 k'_2 k_2, \\
\delta_{\psi vy} &= -\frac{1}{\delta_{\text{sat}}^2} 2k'_1 k'_2 \frac{k_1}{x_d} - 2\frac{T^2 k_1}{x_d^3}, \\
\delta_{\psi ry} &= -\frac{1}{\delta_{\text{sat}}^2} 2k'_1 k_2 \frac{k_1}{x_d}, \\
\delta_{vry} &= -\frac{1}{\delta_{\text{sat}}^2} 2k'_2 k_2 \frac{k_1}{x_d},
\end{aligned}$$

$$\begin{aligned}
\delta_{\psi\psi\psi} &= -\frac{1}{\delta_{\text{sat}}^2} \frac{1}{3} (k'_1)^3 + \frac{k_1 T}{6x_d} + \frac{k_1 T^3}{3x_d^3} , \\
\delta_{vvv} &= -\frac{1}{\delta_{\text{sat}}^2} \frac{1}{3} (k'_2)^3 + \frac{k_1 T^3}{3x_d^3} , \\
\delta_{rrr} &= -\frac{1}{\delta_{\text{sat}}^2} \frac{1}{3} k_2^3 , \\
\delta_{yyy} &= -\frac{1}{\delta_{\text{sat}}^2} \frac{1}{3} \frac{k_1^3}{x_d^3} - \frac{2k_1}{3x_d^3} ,
\end{aligned}$$

and

$$\begin{aligned}
k'_1 &= k_1 - k_1 \frac{T}{x_d} , \\
k'_2 &= -k_1 \frac{T}{x_d} .
\end{aligned}$$

### C. NONLINEAR COEFFICIENT $\mathcal{K}$

If we introduce polar coordinates in the form,

$$z_1 = R \cos \theta , \quad z_2 = R \sin \theta , \quad (58)$$

we can use (54) and (55) to produce an equation describing the rate of change of the radial coordinate  $R$ ,

$$\dot{R} = \alpha' \varepsilon R + \mathcal{P}(\theta) R^3 , \quad (59)$$

and a similar equation in the rate of change of the angular coordinate  $\theta$ ,

$$\dot{\theta} = \omega_0 + \omega' \varepsilon + \mathcal{Q}(\theta) R^2 . \quad (60)$$

The system of equations (59) and (60) contains two variables, one of which,  $R$ , is slowly varying in time, whereas the other one,  $\theta$  is a fast variable. Then,

equation (59) can be averaged over one cycle in  $\theta$  to produce an equation with constant coefficients and similar stability properties,

$$\dot{R} = \alpha' \varepsilon R + \mathcal{K} R^3, \quad (61)$$

where,

$$\mathcal{K} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(\theta) d\theta = \frac{1}{8} (3r_{11} + r_{13} + r_{22} + 3r_{24}). \quad (62)$$

Nontrivial equilibrium solutions of (61) correspond to limit cycles in the original system. The nontrivial equilibrium of (61),  $R_0$ , is given by,

$$R_0 = \sqrt{-\frac{\alpha' \varepsilon}{\mathcal{K}}}. \quad (63)$$

In our problem, since the trivial equilibrium gains its stability for  $x_d > x_{d\text{critical}}$ , the coefficient  $\alpha'$  is always negative. Therefore, it can be seen from (63) that a limit cycle will exist provided  $\mathcal{K}$  and  $\varepsilon$  have the same sign. The stability properties of this limit cycle can be determined by linearization of (61) around  $R_0$ . The linearized system is,

$$\dot{R} = -2\alpha' \varepsilon (R - R_0), \quad (64)$$

and its eigenvalue is,

$$\beta = -2\alpha' \varepsilon. \quad (65)$$

We can see that the Floquet exponent (65) is negative if  $\varepsilon$  is negative, which means that  $\mathcal{K}$  must be negative. Therefore, location and stability of limit cycles depends entirely on the cubic coefficient  $\mathcal{K}$  which is computable from (62). We can summarize our findings as,

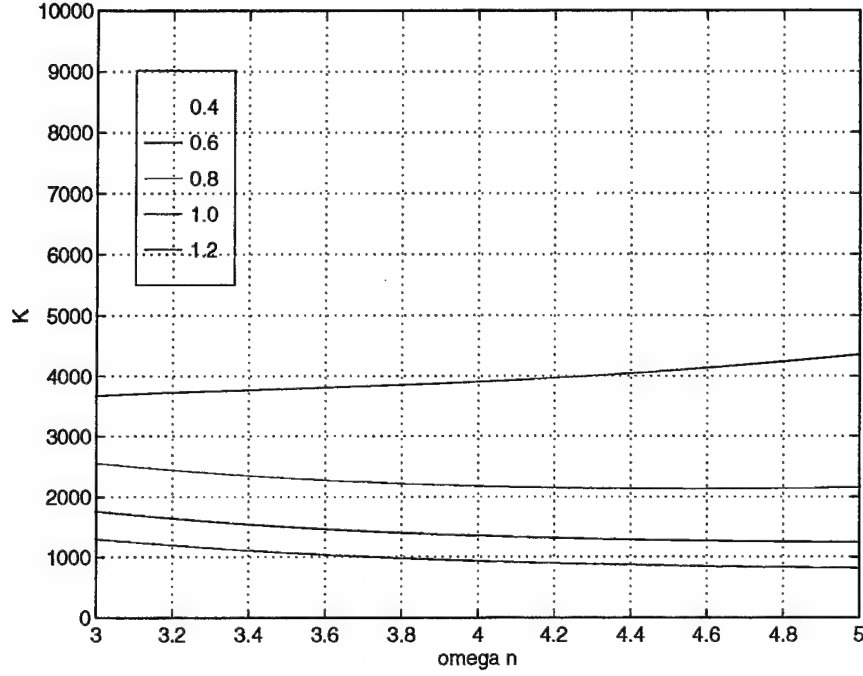


Figure 7: Nonlinear coefficient  $\mathcal{K}$  versus  $\omega_n$  for  $\delta_{\text{sat}} = 0.4$  and various values of  $\zeta$

- If  $\mathcal{K} < 0$  then limit cycles exist for  $\varepsilon < 0$  ( $x_d < x_{d_{\text{critical}}}$ ) and they are stable.
- If  $\mathcal{K} > 0$  then limit cycles exist for  $\varepsilon > 0$  ( $x_d > x_{d_{\text{critical}}}$ ) and they are unstable.

#### D. RESULTS AND DISCUSSION

Typical results in terms of the nonlinear stability coefficient  $\mathcal{K}$  are shown in Figures 7 through 10. The general conclusion from this series of graphs is that the bifurcations are all strongly subcritical. This means that any linearized stability results should be viewed with extreme caution. Limit cycles exist

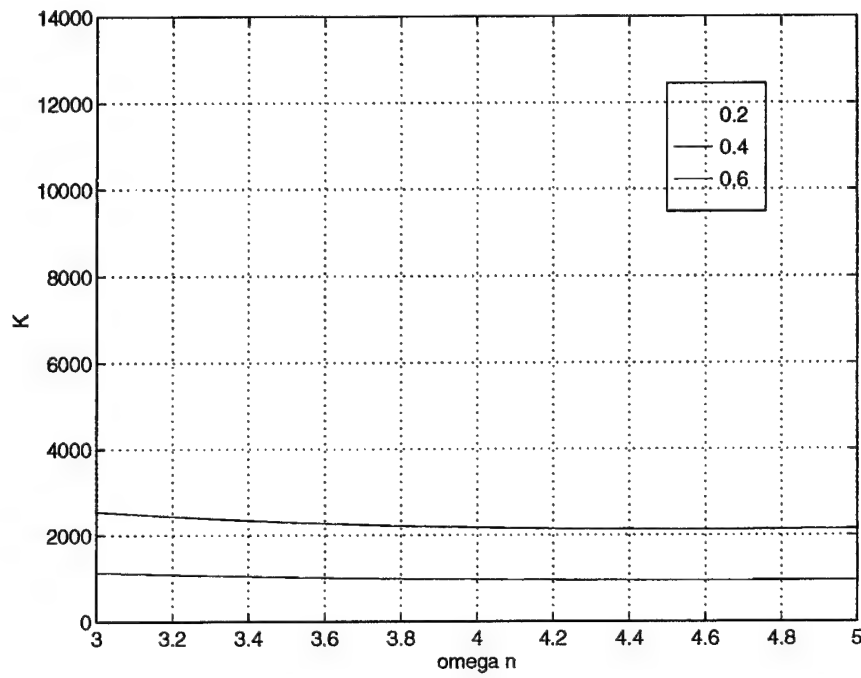


Figure 8: Nonlinear coefficient  $\mathcal{K}$  versus  $\omega_n$  for  $\zeta = 0.8$  and various values of  $\delta_{\text{sat}}$

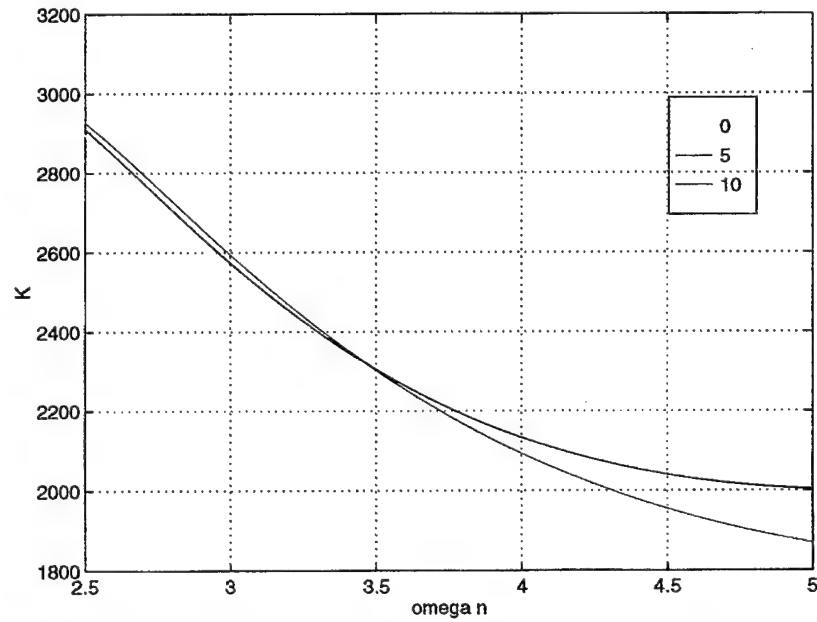


Figure 9: Nonlinear coefficient  $\mathcal{K}$  versus  $\omega_n$  for  $\zeta = 0.8$ ,  $\delta_{\text{sat}} = 0.4$ ,  $T_1 = 0$ , and various values of  $T_2$  (sec)

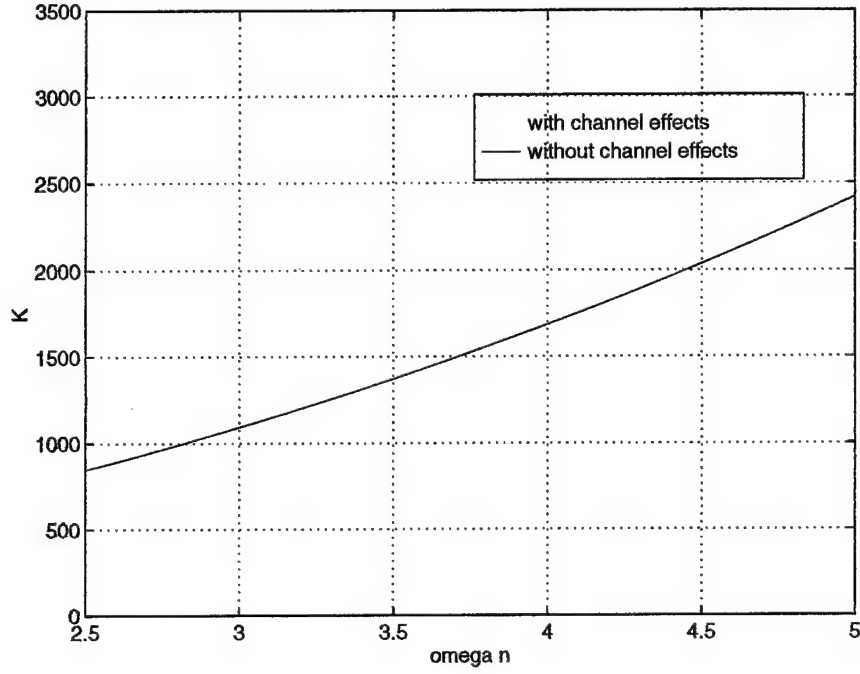


Figure 10: Nonlinear coefficient  $\mathcal{K}$  versus  $\omega_n$  with and without channel effects

before stability in the linear sense is lost and a self sustained oscillation in the system may develop as a result of an external disturbance even if the nominal equilibrium state is still stable. The bifurcations become more subcritical; i.e.,  $\mathcal{K}$  is more positive, for smaller values of  $\zeta$ , as Figure 7 shows. Figure 8 demonstrates the effect that the saturation limit  $\delta_{\text{sat}}$  has on the value of  $\mathcal{K}$ . Higher values of  $\delta_{\text{sat}}$ , although are not related to the critical value of  $x_d$  result in significant changes in the value of  $\mathcal{K}$ . The general trend is that higher  $\delta_{\text{sat}}$  results in less subcritical bifurcations as evidenced by the overall decrease of  $\mathcal{K}$ . Figure 9 shows the effect of non-zero time lag on the nonlinear stability coefficient. It can be seen that, like the linear stability results, the effect is minimal. Finally, Figure 10 shows the canal effect on  $\mathcal{K}$ . This figure was

produced for zero time lag,  $\zeta = 0.8$ , and  $\delta_{\text{sat}} = 0.4$ . It can be seen that the existence of a canal causes the bifurcations to be much more subcritical than the open water case. This demonstrates the severe destabilizing effect that the canal introduces in both the linearized and the nonlinear analysis.

## V. CONCLUSIONS AND RECOMMENDATIONS

This thesis presented a comprehensive study of linear and nonlinear stability properties of straightline motion of surface ships in confined waters. The classical maneuvering equations of motion incorporating canal suction effects, were coupled with appropriate navigation, guidance, and control laws in order to mimic the helmsman's behavior. The main conclusions from this study can be summarized as follows:

1. There exists a critical preview distance for straightline positional stability. For values less than the critical distance, the system is unstable.
2. Including canal effects, the critical preview distance may be an order of magnitude higher than in open water. If the same preview distance is to be used in both cases, the corresponding control law for canal maneuvering must be considerably more responsive than in open waters.
3. The critical preview distance is monotonically decreasing for increasing control law responsiveness. This means that in order to accommodate smaller values of the preview distance, more responsive control laws are required .
4. Physically realizable time lags do not seem to have a significant effect on the values of the preview distance.
5. As the preview distance becomes less than its critical value, one pair of

complex conjugate eigenvalues of the linearized system matrix crosses the imaginary axis. This corresponds to a bifurcation to periodic solutions (Hopf bifurcation) and the system exhibits oscillatory behavior, also known as limit cycles.

6. Higher order approximations in the equations of motion were utilized in order to assess the stability of the resulting limit cycles. It was found that in all cases, the limit cycles were unstable. This has the following implications:

- (a) It is possible for the system to lose its stability even *before* the critical preview distance is crossed. This means that all linearized stability analysis results should be viewed with extreme caution.
- (b) As the critical preview distance is crossed, it is expected that the system will develop limit cycles of large amplitude. This clearly presents a dangerous situation which should be avoided in practice, by appropriate changes in the design parameters.

Recommendations for further research include the following:

1. Simulation studies in order to verify the limit cycle behavior.
2. Study of different ship characteristics and canal geometry.

## APPENDIX

The following is a list and description of the computer programs used in this thesis. The programs are written in FORTRAN. Complete printouts of the programs follow. Standard eigenvalue/eigenvector numerical analysis subroutines are required for all programs.

- **THESIS1.FOR**

Calculation of critical  $x_d$ . First order approximation for time lag  $T_2$ .

- **THESIS2.FOR**

Calculation of critical  $x_d$ . Second order approximation for time lag  $T_2$ .

- **THESIS3.FOR**

Calculation of critical  $x_d$ . Third order approximation for time lag  $T_2$ .

- **THESIS4.FOR**

Calculation of critical  $x_d$ . First order approximation for time lag  $T_1$ .

- **THESIS5.FOR**

Calculation of critical  $x_d$ . First order approximation for time lags  $T_1$  and  $T_2$ .

- **HOPF.FOR**

Calculation of the nonlinear cubic stability coefficient  $\mathcal{K}$ . Requires the output of one of the previous programs as its input.

```

C   PROGRAM THESIS1.FOR (Time Delay-1st Order Approx. T2)
C   BIFURCATION ANALYSIS
C
C   PROGRAM THESIS1
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   DOUBLE PRECISION K1,K2,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
C   &      NRDOT,NVDOT
C   DIMENSION A(4,4),FV1(4),IV1(4),ZZZ(4,4),WR(4),WI(4)
C
C   OPEN (11,FILE='BIF1.RES')
C   OPEN (12,FILE='BIF2.RES')
C   OPEN (13,FILE='BIF3.RES')
C   open (15,file='eig.res1')
C
C   Vehicle Parameters:
C   IZ      =0.0
C   L       =528
C   RHO     =1.94
C   XG      =0.0
C   MASS    =0.0088
C   U       =1.0
C
C   YRDOT = 0.00000
C   YVDOT =-0.00912
C   YR     =+0.00456
C   YV     =-0.01434
C   YPSI   = 0.01400
C   YY     = 0.02000
C   YDELTA= 0.00278
C   NRDOT  =-0.00115
C   NVDOT  = 0.00000
C   NR     =-0.00296
C   NV     =-0.00460
C   NPSI   = 0.01000
C   NY     =-0.00250
C   NDELTA=-0.00139
C   WRITE (*,1001)
C   READ  (*,*)      WNMIN,WNMAX,IWN
C   WRITE (*,1002)
C   READ  (*,*)      XDMIN,XDMAX,IXD
C   WRITE (*,1003)
C   READ  (*,*)      ZETA
C   WRITE(*,1100)
C   READ (*,*) TL
C   TL=TL*U/L
C
C   DVR =(IZ-NRDOT)*(MASS-YVDOT)-
C   &      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

```

```

AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
AA13=((IZ-NRDOT)*(YR-MASS)+
&      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
AA23=((NVDOT-MASS*XG)*(YR-MASS)+
&      (MASS-YVDOT)*(NR-MASS*XG))/DVR
AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR
BB1=((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
BB2=((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR
C
ANOM=AA23
BNOM=BB2
CNOM=AA21
EPS  =1.D-5
ILMAX=1500
C
DO 1 I=1,IWN
  WRITE (*,2001) I,IWN
  WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
  K1=-(WN**2.DO/BNOM)-(CNOM/BNOM)
  K2=-(ANOM+2.DO*ZETA*WN)/BNOM
DO 2 J=1,IXD
  XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
  CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
&  AA24,BB1,BB2,A,TL)
  CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
  CALL DSTABL(DEOS,WR,WI,FREQ)
  write(15,*)DEOS,XD,WN
C
  IF (J.GT.1) GO TO 10
  DEOS00=DEOS
  XD00 =XD
  LL=0
  GO TO 2
10  DEOSNN=DEOS
  XDNN =XD
  PR=DEOSNN*DEOS00
  IF (PR.GT.0.DO) GO TO 3
  LL=LL+1
  IF (LL.GT.3) STOP 1000
  IL=0
  XD0=XD00
  XDNN=XDNN
  DEOS0=DEOS00

```

```

        DEOSN=DEOSNN
6      XDL=XDO
        XDR=XDN
        DEOSL=DEOSO
        DEOSR=DEOSN
        XD=(XDL+XDR)/2.D0
C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
&      AA22,AA23,AA24,BB1,BB2,A,TL)
        CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        DEOSM=DEOS
        XDM=XD
        PRL=DEOSL*DEOSM
        PRR=DEOSR*DEOSM
        IF (PRL.GT.0.D0) GO TO 5
        XDO=XDL
        XDN=XDM
        DEOSO=DEOSL
        DEOSN=DEOSM
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDL-XDM)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
        GO TO 4
5      IF (PRR.GT.0.D0) STOP 3200
        XDO=XDM
        XDN=XDR
        DEOSO=DEOSM
        DEOSN=DEOSR
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDM-XDR)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
4      LLL=10+LL
        WRITE (LLL,*) XD,WN
3      XDOO=XDNN
        DEOSOO=DEOSNN
2      CONTINUE
1      CONTINUE
C
1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003 FORMAT (' ENTER DAMPING RATIO')
1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
2001 FORMAT (2I5)

```

```

END
C
SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION WR(4),WI(4)
DEOS=-1.0D+20
DO 1 I=1,4
  IF (WR(I).LT.DEOS) GO TO 1
  DEOS=WR(I)
  IJ=I
1 CONTINUE
OMEGA=WI(IJ)
OMEGA=DABS(OMEGA)
RETURN
END
C
SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
& AA22,AA23,AA24,BB1,BB2,A,TL)
C
C FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation
C
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DOUBLE PRECISION K1,K2
DIMENSION A(4,4)
A(1,1)=0.0D0
A(1,2)=0.0D0
A(1,3)=1.0D0
A(1,4)=0.0D0
A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
A(2,2)=AA12-BB1*K1*TL/XD
A(2,3)=AA13+BB1*K2
A(2,4)=AA14+BB1*K1/XD
A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
A(3,2)=AA22-BB2*K1*TL/XD
A(3,3)=AA23+BB2*K2
A(3,4)=AA24+BB2*K1/XD
A(4,1)=1.0D0
A(4,2)=1.0D0
A(4,3)=0.0D0
A(4,4)=0.0D0
RETURN
END

```

```

C   PROGRAM THESIS2.FOR (Time Delay-2nd Order Approx T2)
C   BIFURCATION ANALYSIS
C
C   PROGRAM THESIS2
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   DOUBLE PRECISION K1,K2,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
C   & NRDOT,NVDOT
C   DIMENSION A(4,4),FV1(4),IV1(4),ZZZ(4,4),WR(4),WI(4)
C
C   OPEN (11,FILE='BIF1.RES')
C   OPEN (12,FILE='BIF2.RES')
C   OPEN (13,FILE='BIF3.RES')
C
C   open (15,file='eig.res1')
C
C   Vehicle Parameters:
C   IZ    =0.0
C   L     =528
C   RHO   =1.94
C   XG    =0.0
C   MASS  =0.0088
C   U     =1.0
C
C   YRDOT = 0.00000
C   YVDOT =-0.00912
C   YR    =+0.00456
C   YV    =-0.01434
C
C   YPSI  = 0.01400
C   YY    = 0.02000
C   YDELTA= 0.00278
C
C   NRDOT =-0.00115
C   NVDOT = 0.00000
C   NR    =-0.00296
C   NV    =-0.00460
C
C   NPSI  = 0.01000
C   NY    =-0.00250
C
C   NDELTA=-0.00139
C   WRITE (*,1001)
C
C   READ  (*,*)    WNMIN,WNMAX,IWN
C   WRITE (*,1002)
C   READ  (*,*)    XDMIN,XDMAX,IXD
C   WRITE (*,1003)

```

```

      READ (*,*)      ZETA
      WRITE(*,1100)
      READ (*,*) TL
      TL=TL*U/L

C
      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
&      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
&      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
&      (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

      BB1=((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2=((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

C
      ANOM=AA23
      BNOM=BB2
      CNOM=AA21

      EPS  =1.D-5
      ILMAX=1500

C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-(WN**2.DO/BNOM)-(CNOM/BNOM)

        K2=-(ANOM+2.DO*ZETA*WN)/BNOM

      DO 2 J=1,IXD
        XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)

C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
&      AA24,BB1,BB2,A,TL)
        CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
        CALL DSTABL(DEOS,WR,WI,FREQ)
        write(15,*)DEOS,XD,WN

C
        IF (J.GT.1) GO TO 10
        DEOS00=DEOS
        XD00  =XD

```

```

LL=0
GO TO 2
10 DEOSNN=DEOS
   XDNN =XD
   PR=DEOSNN*DEOS00
   IF (PR.GT.0.D0) GO TO 3
   LL=LL+1
   IF (LL.GT.3) STOP 1000
   IL=0
   XDO=XDOO
   XDN=XDNN
   DEOSO=DEOS00
   DEOSN=DEOSNN
6   XDL=XDO
   XDR=XDN
   DEOSL=DEOSO
   DEOSR=DEOSN
   XD=(XDL+XDR)/2.D0
C
   CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
&   AA22,AA23,AA24,BB1,BB2,A,TL)
   CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
   CALL DSTABL(DEOS,WR,WI,FREQ)
   DEOSM=DEOS
   XDM=XD
   PRL=DEOSL*DEOSM
   PRR=DEOSR*DEOSM
   IF (PRL.GT.0.D0) GO TO 5
   XDO=XDL
   XDN=XDM
   DEOSO=DEOSL
   DEOSN=DEOSM
   IL=IL+1
   IF (IL.GT.ILMAX) STOP 3100
   DIF=DABS(XDL-XDM)
   IF (DIF.GT.EPS) GO TO 6
   XD=XDM
   GO TO 4
5   IF (PRR.GT.0.D0) STOP 3200
   XDO=XDM
   XDN=XDR
   DEOSO=DEOSM
   DEOSN=DEOSR
   IL=IL+1
   IF (IL.GT.ILMAX) STOP 3100
   DIF=DABS(XDM-XDR)
   IF (DIF.GT.EPS) GO TO 6
   XD=XDM

```

```

4     LLL=10+LL
      WRITE (LLL,*) XD,WN
3     XD00=XDNN
      DEOS00=DEOSNN
2     CONTINUE
1     CONTINUE
C
1001  FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002  FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003  FORMAT (' ENTER DAMPING RATIO')
1100  FORMAT (' ENTER TIME LAG TL (dimensional)')
2001  FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C     EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
1     CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
&      AA22,AA23,AA24,BB1,BB2,A,TL)
C
C     FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation)
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(2,1)=(AA11*XD+BB1*K1*XD-BB1*K1*TL)/(XD-0.50D0*BB1*K1*TL*TL)
      A(2,2)=(AA12*XD-BB1*K1*TL)/(XD-0.50D0*BB1*K1*TL*TL)
      A(2,3)=(AA13*XD+BB1*K2*XD+0.50D0*BB1*K1*TL*TL)/
&      (XD-0.50D0*BB1*K1*TL*TL)
      A(2,4)=(AA14*XD+BB1*K1)/(XD-0.50D0*BB1*K1*TL*TL)

```

```

A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,1)
A(3,2)=AA22-BB2*K1*TL/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,2)
A(3,3)=AA23+BB2*K2+(BB2*K1*TL*TL/(2.0D0*XD))+
&      (BB2*K1*TL*TL/(2.0*XD))* A(2,3)
A(3,4)=AA24+BB2*K1/XD+(BB2*K1*TL*TL/(2.0D0*XD))*A(2,4)
A(4,1)=1.0D0
A(4,2)=1.0D0
A(4,3)=0.0D0
A(4,4)=0.0D0
RETURN
END

```

```

C   PROGRAM THESIS3.FOR (Time Delay-3rd Order Approx T2)
C   BIFURCATION ANALYSIS
C
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
&   NRDOT,NVDOT
C   DIMENSION A(5,5),B(5,5),FV1(5),IV1(5),ZZZ(5,5),ALFR(5),
&   ALFI(5),BETA(5),WR(5),WI(5)
C
C   OPEN (11,FILE='BIF1.RES')
C   OPEN (12,FILE='BIF2.RES')
C   OPEN (13,FILE='BIF3.RES')
C
C   Vehicle Parameters:
C   IZ   =0.0
C   L    =528
C   RHO  =1.94
C   XG   =0.0
C   MASS =0.0088
C   U    =1.0
C
C   YRDOT = 0.00000
C   YVDOT =-0.00912
C   YR    =+0.00456
C   YV    =-0.01434
C   YPSI  = 0.01400
C   YY    = 0.02000
C   YDELTA= 0.00278
C   NRDOT =-0.00115
C   NVDOT = 0.00000
C   NR    =-0.00296
C   NV    =-0.00460
C   NPSI  = 0.01000
C   NY    =-0.00250
C   NDELTA=-0.00139
C   WRITE (*,1001)
C   READ  (*,*)   WNMIN,WNMAX,IWN
C   WRITE (*,1002)
C   READ  (*,*)   XDMIN,XDMAX,IXD
C   WRITE (*,1003)
C   READ  (*,*)   ZETA
C   WRITE(*,1100)
C   READ  (*,*)   TL
C   TL=TL*U/L
C
C   DVR =(IZ-NRDOT)*(MASS-YVDOT)-
&   (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

```

```

AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
AA13=((IZ-NRDOT)*(YR-MASS)+
& (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
AA23=((NVDOT-MASS*XG)*(YR-MASS)+
& (MASS-YVDOT)*(NR-MASS*XG))/DVR
AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

BB1=((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
BB2=((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

ANOM=AA23
BNOM=BB2
CNOM=AA21

EPS =1.D-5
ILMAX=1500

C
DO 1 I=1,IWN
  WRITE (*,2001) I,IWN
  WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
  K1=-(WN**2.DO/BNOM)-(CNOM/BNOM)
  K2=-(ANOM+2.DO*ZETA*WN)/BNOM
  DO 2 J=1,IXD
    XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)
C
    CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
& AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL)
    CALL RGG(5,5,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
    DO 11 IJE=1,5
      WR(IJE)=ALFR(IJE)/BETA(IJE)
      WI(IJE)=ALFI(IJE)/BETA(IJE)
11  CONTINUE
    CALL DSTABL(DEOS,WR,WI,FREQ)
C
    IF (J.GT.1) GO TO 10
    DEOS00=DEOS
    XD00 =XD
    LL=0
    GO TO 2
10  DEOSNN=DEOS
    XDNN =XD
    PR=DEOSNN*DEOS00
    IF (PR.GT.0.DO) GO TO 3
    LL=LL+1

```

```

        IF (LL.GT.3) STOP 1000
        IL=0
        XDO=XDOO
        XDN=XDNN
        DEOSO=DEOSOO
        DEOSN=DEOSNN
6      XDL=XDO
        XDR=XDN
        DEOSL=DEOSO
        DEOSR=DEOSN
        XD=(XDL+XDR)/2.DO
C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
&          AA23,AA24,BB1,BB2,A,B,TL)
        CALL RGG(5,5,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
        DO 12 IJE=1,5
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
12     CONTINUE
        CALL DSTABL(DEOS,WR,WI,FREQ)
C
        DEOSM=DEOS
        XDM=XD
        PRL=DEOSL*DEOSM
        PRR=DEOSR*DEOSM
        IF (PRL.GT.0.DO) GO TO 5
        XDO=XDL
        XDN=XDM
        DEOSO=DEOSL
        DEOSN=DEOSM
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDL-XDM)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
        GO TO 4
5      IF (PRR.GT.0.DO) STOP 3200
        XDO=XDM
        XDN=XDR
        DEOSO=DEOSM
        DEOSN=DEOSR
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDM-XDR)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
4      LLL=10+LL
        WRITE (LLL,*) XD,WN

```

```

3      XD00=XDNN
      DEOS00=DEOSNN
2      CONTINUE
1      CONTINUE
C
1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003 FORMAT (' ENTER DAMPING RATIO')
1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
2001 FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(5),WI(5)
      DEOS=-1.0D+20
      DO 1 I=1,5
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
1      CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
&      ,BB1,BB2,A,B,TL)

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(5,5),B(5,5)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(1,5)=0.0D0
      A(2,1)=0.0D0
      A(2,2)=0.0D0
      A(2,3)=0.0D0
      A(2,4)=0.0D0
      A(2,5)=1.0D0
      A(3,1)=AA11-(BB1*K1*TL/XD)+BB1*K1
      A(3,2)=AA12-BB1*K1*TL/XD
      A(3,3)=AA13+BB1*K2+(BB1*K1*TL*TL/(2.D0*XD))
      A(3,4)=AA14+BB1*K1/XD
      A(3,5)=-1.D0+(BB1*K1*TL*TL/(2.D0*XD))
      A(4,1)=1.0D0

```

```

A(4,2)=1.0D0
A(4,3)=0.0D0
A(4,4)=0.0D0
A(4,5)=0.0D0
A(5,1)=AA21+BB2*K1-BB2*K1*TL/XD
A(5,2)=AA22-BB2*K1*TL/XD
A(5,3)=AA23+BB2*K2+(BB2*K1*TL*TL/(2.D0*XD))
A(5,4)=AA24+BB2*K1/XD
A(5,5)=(BB2*K1*TL*TL/(2.D0*XD))

```

C

```

B(1,1)=1.0D0
B(1,2)=0.0D0
B(1,3)=0.0D0
B(1,4)=0.0D0
B(1,5)=0.0D0
B(2,1)=0.0D0
B(2,2)=1.0D0
B(2,3)=0.0D0
B(2,4)=0.0D0
B(2,5)=0.0D0
B(3,1)=0.0D0
B(3,2)=0.0D0
B(3,3)=(BB1*K1*TL*TL*TL/(6.D0*XD))
B(3,4)=0.0D0
B(3,5)=(BB1*K1*TL*TL*TL/(6.D0*XD))
B(4,1)=0.0D0
B(4,2)=0.0D0
B(4,3)=0.0D0
B(4,4)=1.0D0
B(4,5)=0.0D0
B(5,1)=0.0D0
B(5,2)=0.0D0
B(5,3)=1.0D0+(BB2*K1*TL*TL*TL/(6.D0*XD))
B(5,4)=0.0D0
B(5,5)=(BB2*K1*TL*TL*TL/(6.D0*XD))
RETURN
END

```

```

C   PROGRAM THESIS4.FOR (Time Delay-1st Order Approx in delta ,T1)
C   BIFURCATION ANALYSIS
C
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
C   & NRDOT,NVDOT
C   DIMENSION A(4,4),B(4,4),FV1(4),IV1(4),ZZZ(4,4),ALFR(4),
C   & ALFI(4),BETA(4),WR(4),WI(4)
C
C   OPEN (11,FILE='BIF1.RES')
C   OPEN (12,FILE='BIF2.RES')
C   OPEN (13,FILE='BIF3.RES')
C
C   Vehicle Parameters:
C   IZ   =0.0
C   L    =528
C   RHO  =1.94
C   G    =32.2
C   XG   =0.0
C   MASS =0.0088
C   U    =3.0
C
C
C   YRDOT = 0.00000
C   YVDOT =-0.00912
C   YR    =+0.00456
C   YV    =-0.01434
C
C   YPSI  = 0.01400
C   YY    = 0.02000
C
C   YDELTA= 0.00278
C   NRDOT =-0.00115
C   NVDOT = 0.00000
C   NR    =-0.00296
C   NV    =-0.00460
C
C   NPSI  = 0.01000
C   NY    = -0.00250
C
C   NDELTA=-0.00139
C
C   WRITE (*,1001)
C   READ  (*,*) WNMIN,WNMAX,IWN
C   WRITE (*,1002)
C   READ  (*,*) XDMIN,XDMAX,IXD
C   WRITE (*,1003)

```

```

      READ (*,*)      ZETA
      WRITE(*,1100)
      READ (*,*) TL
      TL=TL*U/L

C
      DVR =(IZ-NRDOT)*(MASS-YVDOT)-
&      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

      AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
      AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
      AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
      AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
      AA13=((IZ-NRDOT)*(YR-MASS)+
&      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
      AA23=((NVDOT-MASS*XG)*(YR-MASS)+
&      (MASS-YVDOT)*(NR-MASS*XG))/DVR
      AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
      AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

      BB1=((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
      BB2=((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

      ANOM=AA23
      BNOM=BB2

      EPS  =1.D-5
      ILMAX=1500

C
      DO 1 I=1,IWN
        WRITE (*,2001) I,IWN
        WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
        K1=-WN**2.DO/BNOM
        K2=-(ANOM+2.DO*ZETA*WN)/BNOM
        DO 2 J=1,IXD
          XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)

C
          CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
&          AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL)
          CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
          DO 11 IJE=1,4
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
11          CONTINUE
          CALL DSTABL(DEOS,WR,WI,FREQ)

C
          IF (J.GT.1) GO TO 10
          DEOS00=DEOS
          XD00  =XD

```

```

LL=0
GO TO 2
10 DEOSNN=DEOS
XDNN =XD
PR=DEOSNN*DEOS00
IF (PR.GT.0.D0) GO TO 3
LL=LL+1
IF (LL.GT.3) STOP 1000
IL=0
XDO=XDOO
XDN=XDNN
DEOS0=DEOS00
DEOSN=DEOSNN
6 XDL=XDO
XDR=XDN
DEOSL=DEOS0
DEOSR=DEOSN
XD=(XDL+XDR)/2.D0
C
CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
& AA23,AA24,BB1,BB2,A,B,TL)
CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
DO 12 IJE=1,4
WR(IJE)=ALFR(IJE)/BETA(IJE)
WI(IJE)=ALFI(IJE)/BETA(IJE)
12 CONTINUE
CALL DSTABL(DEOS,WR,WI,FREQ)
C
DEOSM=DEOS
XDM=XD
PRL=DEOSL*DEOSM
PRR=DEOSR*DEOSM
IF (PRL.GT.0.D0) GO TO 5
XDO=XDL
XDN=XDM
DEOS0=DEOSL
DEOSN=DEOSM
IL=IL+1
IF (IL.GT.ILMAX) STOP 3100
DIF=DABS(XDL-XDM)
IF (DIF.GT.EPS) GO TO 6
XD=XDM
GO TO 4
5 IF (PRR.GT.0.D0) STOP 3200
XDO=XDM
XDN=XDR
DEOS0=DEOSM
DEOSN=DEOSR

```

```

        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDM-XDR)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
4       LLL=10+LL
        WRITE (LLL,*) XD,WN
3       XDOO=XDNN
        DEOSOO=DEOSNN
2       CONTINUE
1       CONTINUE
C
1001  FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002  FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003  FORMAT (' ENTER DAMPING RATIO')
1100  FORMAT (' ENTER TIME LAG TL (dimensional)')
2001  FORMAT (2I5)
      END
C
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
1     CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
&          ,BB1,BB2,A,B,TL)

      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4),B(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0

      A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
      A(2,2)=AA12-BB1*K1*TL/XD
      A(2,3)=AA13+BB1*K2-BB1*K1*TL
      A(2,4)=AA14+BB1*K1/XD

```

```
A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
A(3,2)=AA22-BB2*K1*TL/XD
A(3,3)=AA23+BB2*K2-BB2*K1*TL
A(3,4)=AA24+BB2*K1/XD
```

```
A(4,1)=1.0D0
A(4,2)=1.0D0
A(4,3)=0.0D0
A(4,4)=0.0D0
```

C

```
B(1,1)=1.0D0
B(1,2)=0.0D0
B(1,3)=0.0D0
B(1,4)=0.0D0
```

```
B(2,1)=0.0D0
B(2,2)=1.0D0
B(2,3)=BB1*K2*TL
B(2,4)=0.0D0
```

```
B(3,1)=0.0D0
B(3,2)=0.0D0
B(3,3)=1.0D0+(BB2*K2*TL)
B(3,4)=0.0D0
```

```
B(4,1)=0.0D0
B(4,2)=0.0D0
B(4,3)=0.0D0
B(4,4)=1.0D0
```

```
RETURN
END
```

```

C   PROGRAM THESIS5.FOR (Time Delay-1st Order Approx in delta & y ie. in T1 & T2))
C   BIFURCATION ANALYSIS
C
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C   DOUBLE PRECISION K1,K2,L,NR,NV,IZ,MASS,NDELTA,NPSI,NY,
C   &                   NRDOT,NVDOT
C   DIMENSION A(4,4),B(4,4),FV1(4),IV1(4),ZZZ(4,4),ALFR(4),
C   &                   ALFI(4),BETA(4),WR(4),WI(4)
C
C   OPEN (11,FILE='BIF1.RES')
C   OPEN (12,FILE='BIF2.RES')
C   OPEN (13,FILE='BIF3.RES')
C
C   Vehicle Parameters:
C   IZ   =0.0
C   L    =528
C   RHO  =1.94
C   G    =32.2
C   XG   =0.0
C   MASS =0.0088
C   U    =3.0
C
C
C   YRDOT = 0.00000
C   YVDOT =-0.00912
C   YR    =+0.00456
C   YV    =-0.01434
C
C   YPSI  = 0.01400
C   YY    = 0.02000
C
C   YDELTA= 0.00278
C   NRDOT =-0.00115
C   NVDOT = 0.00000
C   NR    =-0.00296
C   NV    =-0.00460
C
C   NPSI  = 0.01000
C   NY    = -0.00250
C   NDELTA= -0.00139
C
C   WRITE (*,1001)
C   READ  (*,*)    WNMIN,WNMAX,IWN
C   WRITE (*,1002)
C   READ  (*,*)    XDMIN,XDMAX,IXD
C   WRITE (*,1003)
C   READ  (*,*)    ZETA

```

```

WRITE(*,1100)
READ (*,*)      TL1
TL1=TL1*U/L
WRITE(*,1101)
READ (*,*)      TL2
TL2=TL2*U/L

C
DVR =(IZ-NRDOT)*(MASS-YVDOT)-
&      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)

AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
AA21=((NVDOT-MASS*XG)*YPSI+(MASS-YVDOT)*NPSI)/DVR
AA22=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DVR
AA13=((IZ-NRDOT)*(YR-MASS)+
&      (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
AA23=((NVDOT-MASS*XG)*(YR-MASS)+
&      (MASS-YVDOT)*(NR-MASS*XG))/DVR
AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
AA24=((NVDOT-MASS*XG)*YY+(MASS-YVDOT)*NY)/DVR

BB1=((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
BB2=((MASS-YVDOT)*NDELTA-(MASS*XG-NVDOT)*YDELTA)/DVR

ANOM=AA23
BNOM=BB2
CNOM=AA21

EPS  =1.D-5
ILMAX=1500

C
DO 1 I=1,IWN
  WRITE (*,2001) I,IWN
  WN=WNMIN+(I-1)*(WNMAX-WNMIN)/(IWN-1)
  K1=-(WN**2.DO/BNOM)-(CNOM/BNOM)
  K2=-(ANOM+2.DO*ZETA*WN)/BNOM
  DO 2 J=1,IXD
    XD=XDMIN+(J-1)*(XDMAX-XDMIN)/(IXD-1)

C
    CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,
&      AA21,AA22,AA23,AA24,BB1,BB2,A,B,TL1,TL2)
    CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
    DO 11 IJE=1,4
      WR(IJE)=ALFR(IJE)/BETA(IJE)
      WI(IJE)=ALFI(IJE)/BETA(IJE)
11  CONTINUE
    CALL DSTABL(DEOS,WR,WI,FREQ)

C

```

```

        IF (J.GT.1) GO TO 10
        DEOS00=DEOS
        XD00 =XD
        LL=0
        GO TO 2
10      DEOSNN=DEOS
        XDNN =XD
        PR=DEOSNN*DEOS00
        IF (PR.GT.0.D0) GO TO 3
        LL=LL+1
        IF (LL.GT.3) STOP 1000
        IL=0
        XD0=XD00
        XDN=XDNN
        DEOS0=DEOS00
        DEOSN=DEOSNN
6      XDL=XD0
        XDR=XDN
        DEOSL=DEOS0
        DEOSR=DEOSN
        XD=(XDL+XDR)/2.D0
C
        CALL LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,
&          AA23,AA24,BB1,BB2,A,B,TL1,TL2)
        CALL RGG(4,4,A,B,ALFR,ALFI,BETA,0,ZZZ,IERR)
        DO 12 IJE=1,4
            WR(IJE)=ALFR(IJE)/BETA(IJE)
            WI(IJE)=ALFI(IJE)/BETA(IJE)
12      CONTINUE
        CALL DSTABL(DEOS,WR,WI,FREQ)
C
        DEOSM=DEOS
        XDM=XD
        PRL=DEOSL*DEOSM
        PRR=DEOSR*DEOSM
        IF (PRL.GT.0.D0) GO TO 5
        XD0=XDL
        XDN=XDM
        DEOS0=DEOSL
        DEOSN=DEOSM
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDL-XDM)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
        GO TO 4
5      IF (PRR.GT.0.D0) STOP 3200
        XD0=XDM

```

```

        XDN=XDR
        DEOSO=DEOSM
        DEOSN=DEOSR
        IL=IL+1
        IF (IL.GT.ILMAX) STOP 3100
        DIF=DABS(XDM-XDR)
        IF (DIF.GT.EPS) GO TO 6
        XD=XDM
4       LLL=10+LL
        WRITE (LLL,*) XD,WN
3       XD00=XDNN
        DEOS00=DEOSNN
2       CONTINUE
1       CONTINUE
C
1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003 FORMAT (' ENTER DAMPING RATIO')
1100 FORMAT (' ENTER TIME LAG TL1 (dimensional)')
1101 FORMAT (' ENTER TIME LAG TL2 (dimensional)')
2001 FORMAT (2I5)
        END
C
        SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DIMENSION WR(4),WI(4)
        DEOS=-1.0D+20
        DO 1 I=1,4
            IF (WR(I).LT.DEOS) GO TO 1
            DEOS=WR(I)
            IJ=I
1       CONTINUE
        OMEGA=WI(IJ)
        OMEGA=DABS(OMEGA)
        RETURN
        END
C
        SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,AA24
&                ,BB1,BB2,A,B,TL1,TL2)

        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        DOUBLE PRECISION K1,K2
        DIMENSION A(4,4),B(4,4)
        A(1,1)=0.0D0
        A(1,2)=0.0D0
        A(1,3)=1.0D0
        A(1,4)=0.0D0

```

```

A(2,1)=AA11+BB1*K1-BB1*K1*TL2/XD-BB1*K1*TL1/XD
A(2,2)=AA12-BB1*K1*TL2/XD-BB1*K1*TL1/XD
A(2,3)=AA13+BB1*K2-BB1*K1*TL1+BB1*K1*TL1*TL2/XD
A(2,4)=AA14+BB1*K1/XD

```

```

A(3,1)=AA21+BB2*K1-BB2*K1*TL2/XD-BB2*K1*TL1/XD
A(3,2)=AA22-BB2*K1*TL2/XD-BB2*K1*TL1/XD
A(3,3)=AA23+BB2*K2-BB2*K1*TL1+BB2*K1*TL1*TL2/XD
A(3,4)=AA24+BB2*K1/XD

```

```

A(4,1)=1.0D0
A(4,2)=1.0D0
A(4,3)=0.0D0
A(4,4)=0.0D0

```

C

```

B(1,1)=1.0D0
B(1,2)=0.0D0
B(1,3)=0.0D0
B(1,4)=0.0D0

```

```

B(2,1)=0.0D0
B(2,2)=1.0D0-(BB1*K1*TL1*TL2/XD)
B(2,3)=BB1*K2*TL1
B(2,4)=0.0D0

```

```

B(3,1)=0.0D0
B(3,2)=-BB2*K1*TL1*TL2/XD
B(3,3)=1.0D0+(BB2*K2*TL1)
B(3,4)=0.0D0

```

```

B(4,1)=0.0D0
B(4,2)=0.0D0
B(4,3)=0.0D0
B(4,4)=1.0D0

```

```

RETURN
END

```

```

C      PROGRAM HOPF.FOR
C
C      Hopf Bifurcation Analysis
C
C      Third Order Expansions:  First Order Approximation
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2,K3,L,NR,NV,NDELTA,NPSI,NY,IZ,MASS,
&      NRDOT,NVDOT,K1P,K2P,NVVV,NVRR,NRVV,NYYY,NPPP
      DOUBLE PRECISION M11,M12,M13,M14,M21,M22,M23,M24,
1      M31,M32,M33,M34,M41,M42,M43,M44,
2      N11,N12,N13,N14,N21,N22,N23,N24,
3      N31,N32,N33,N34,N41,N42,N43,N44,
4      L21,L22,L23,L24,L31,L32,L33,L34,
5      L41,L42,L43,L44
C
      DIMENSION A(4,4),T(4,4),TINV(4,4),FV1(4),IV1(4),ZZZ(4,4)
      DIMENSION WR(4),WI(4),TSAVE(4,4),TLUD(4,4),IVLUD(4),SVLUD(4)
      DIMENSION ASAVE(4,4)

      OPEN (11,FILE='BIF1.RES')
      OPEN (15,FILE='HOPF.RES')
C
C      Vehicle Parameters:

      IZ      =0.0
      L       =528
      RHO     =1.94
      G       =32.2
      XG      =0.0
      MASS    =0.0088
      U       =1.0
C
      YRDOT   = 0.00000
      YVDOT   =-0.00912
      YR      =+0.00456
      YV      =-0.01434
      YVVV    =-0.15391
      YVRR    =-0.05476
      YRVV    = 0.04608
      YYYY    = 0.46800
      YPPP    = 0.00000
      YPSI    = 0.01400
      YY      = 0.02000
      YDELTA  =+0.00278

```

```

NRDOT =-0.00115
NVDDOT = 0.00000
NR      =-0.00296
NV      =-0.00460
NVVV    =-0.00336
NVRR    = 0.00759
NRVV    =-0.05160
NYYY    = 0.00000
NPPP    = 0.00000
NPSI    = 0.01000
NY      =-0.00250
NDELTA=-0.00139

```

```

WRITE (*,1003)
READ  (*,*)    ZETA
WRITE(*,1100)
READ  (*,*)    TL
TL=TL*U/L
WRITE (*,1006)
READ  (*,*)    D0

```

C

```

DVR =(IZ-NRDOT)*(MASS-YVDDOT)-
&    (MASS*XG-YRDOT)*(MASS*XG-NVDDOT)

AA11=((IZ-NRDOT)*YPSI-(MASS*XG-YRDOT)*NPSI)/DVR
AA12=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DVR
AA21=((NVDDOT-MASS*XG)*YPSI+(MASS-YVDDOT)*NPSI)/DVR
AA22=((MASS-YVDDOT)*NV-(MASS*XG-NVDDOT)*YV)/DVR
AA13=((IZ-NRDOT)*(YR-MASS)+
&    (YRDOT-MASS*XG)*(NR-MASS*XG))/DVR
AA23=((NVDDOT-MASS*XG)*(YR-MASS)+
&    (MASS-YVDDOT)*(NR-MASS*XG))/DVR
AA14=((IZ-NRDOT)*YY+(YRDOT-MASS*XG)*NY)/DVR
AA24=((NVDDOT-MASS*XG)*YY+(MASS-YVDDOT)*NY)/DVR

BB1 =((IZ-NRDOT)*YDELTA-(MASS*XG-YRDOT)*NDELTA)/DVR
BB2 =((MASS-YVDDOT)*NDELTA-(MASS*XG-NVDDOT)*YDELTA)/DVR

```

C

```

ANOM=AA23
BNOM=BB2
CNOM=AA21

```

```

EPS  =1.D-5
ILMAX=1500

```

C

```

IWN=1000

```

```

DO 1 II=1,IWN
  WRITE (*,2001) II
  READ(11,*) XD,WN
  K1=-(WN**2.D0/BNOM)-(CNOM/BNOM)
  K2=-(ANOM+2.D0*ZETA*WN)/BNOM
  K3=K2

C      Start Hopf Bifurcation Analysis
C
C      Evaluate Nonlinear Rudder Expansion Coefficients
C
      K2=0.0D0
      K1P=K1-K1*TL*U/XD
      K2P=K2-K1*TL/XD
C
      DPPV=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K2P
&      + 0.5D0*K1*TL/XD + 3.D0*K1*(TL*U)**3/(3.D0*XD**3)
      DPVV=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K2P*K2P
&      + 3.D0*U*TL*TL*TL*K1/(3.D0*XD**3)
      DPPR=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K3
      DPRR=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K3*K3
      DPPY=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1P*K1/XD
&      - 3.D0*TL*TL*U*U*K1/(3.D0*XD**3)
      DPYY=- (1.D0/(3.D0*D0**2))*3.D0*K1P*K1*K1/(XD**2)
&      + 3.D0*TL*U*K1/(3.D0*XD**3)
      DVVR=- (1.D0/(3.D0*D0**2))*3.D0*K2P*K2P*K3
      DVRR=- (1.D0/(3.D0*D0**2))*3.D0*K2P*K3*K3
      DVVY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K2P*K2P/XD
&      - 3.D0*K1*TL*TL/(3.D0*XD**3)
      DVYY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K1*K2P/(XD**2)
&      + 3.D0*TL*K1/(3.D0*XD**3)
      DRRY=- (1.D0/(3.D0*D0**2))*3.D0*K1*K3*K3/XD
      DRYR=- (1.D0/(3.D0*D0**2))*3.D0*K1*K1*K3/(XD**2)
      DPVR=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K2P*K3
      DPVY=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K1*K2P/XD
&      - 6.D0*TL*TL*U*K1/(3.D0*XD**3)
      DPRY=- (1.D0/(3.D0*D0**2))*6.D0*K1P*K1*K3/XD
      DVRY=- (1.D0/(3.D0*D0**2))*6.D0*K1*K2P*K3/XD
      DPPP=- (1.D0/(3.D0*D0**2))*1.D0*K1P*K1P*K1P
&      + K1*TL*U/(6.D0*XD) + (K1*(TL*U)**3)/(3.D0*XD**3)
      DVVV=- (1.D0/(3.D0*D0**2))*1.D0*K2P*K2P*K2P
&      + K1*(TL**3)/(3.D0*XD**3)
      DRRR=- (1.D0/(3.D0*D0**2))*1.D0*K3*K3*K3
      DYYY=- (1.D0/(3.D0*D0**2))*(K1/XD)**3-K1/(3.D0*XD**3)
&      - K1/(3.D0*XD**3)
C
C      Evaluate Transformation Matrix of Eigenvectors
C

```

```

      CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
&      AA24,BB1,BB2,A,TL)

      DO 11 I=1,4
        DO 12 J=1,4
          ASAVE(I,J)=A(I,J)
12      CONTINUE
11      CONTINUE
      CALL RG(4,4,A,WR,WI,1,ZZZ,IV1,FV1,IERR)
      CALL DSOMEG(IEV,WR,WI,OMEGA,CHECK)
      OMEGAO=OMEGA
      DO 50 I=1,4
        T(I,1)= ZZZ(I,IEV)
        T(I,2)=-ZZZ(I,IEV+1)
50      CONTINUE
        IF (IEV.EQ.1) GO TO 13
        IF (IEV.EQ.2) GO TO 14
        IF (IEV.EQ.3) GO TO 15
        STOP 3004
14      DO 60 I=1,4
        T(I,3)=ZZZ(I,1)
        T(I,4)=ZZZ(I,4)
60      CONTINUE
        GO TO 17
15      DO 70 I=1,4
        T(I,3)=ZZZ(I,1)
        T(I,4)=ZZZ(I,2)
70      CONTINUE
        GO TO 17
13      DO 16 I=1,4
        T(I,3)=ZZZ(I,3)
        T(I,4)=ZZZ(I,4)
16      CONTINUE
17      CONTINUE

C
C      Normalization of the Critical Eigenvector
C
      CALL NORMAL(T)

C
C      Definition of Mij
C
      M11=T(1,1)
      M21=T(2,1)
      M31=T(3,1)
      M41=T(4,1)
      M12=T(1,2)
      M22=T(2,2)

```

M32=T(3,2)  
 M42=T(4,2)  
 M13=T(1,3)  
 M23=T(2,3)  
 M33=T(3,3)  
 M43=T(4,3)  
 M14=T(1,4)  
 M24=T(2,4)  
 M34=T(3,4)  
 M44=T(4,4)

C  
 C  
 C

Definition of Lij

L21= DPPV\*M11\*M11\*M21 + DPVV\*M11\*M21\*M21 + DPPR\*M11\*M11\*M31  
 & + DPRR\*M11\*M31\*M31 + DPPY\*M11\*M11\*M41 + DPYY\*M11\*M41\*M41  
 & + DVVR\*M21\*M21\*M31 + DVRR\*M21\*M31\*M31 + DVVY\*M21\*M21\*M41  
 & + DVYY\*M21\*M41\*M41 + DRRY\*M31\*M31\*M41 + DRYY\*M31\*M41\*M41  
 & + DPVR\*M11\*M21\*M31 + DPVY\*M11\*M21\*M41 + DPRY\*M11\*M41\*M31  
 & + DVRY\*M21\*M41\*M31 + DPPP\*M11\*M11\*M11 + DVVV\*M21\*M21\*M21  
 & + DRRR\*M31\*M31\*M31 + DYYY\*M41\*M41\*M41

C

PPV = M11\*M11\*M22 + 2.0\*M11\*M12\*M21  
 PVV = M12\*M21\*M21 + 2.0\*M11\*M21\*M22  
 PPR = M11\*M11\*M32 + 2.0\*M11\*M12\*M31  
 PRR = M12\*M31\*M31 + 2.0\*M11\*M31\*M32  
 PPY = M11\*M11\*M42 + 2.0\*M11\*M12\*M41  
 PYY = M41\*M41\*M12 + 2.0\*M11\*M41\*M42  
 VVR = M21\*M21\*M32 + 2.0\*M31\*M21\*M22  
 VRR = M22\*M31\*M31 + 2.0\*M31\*M32\*M21  
 VVY = M21\*M21\*M42 + 2.0\*M41\*M21\*M22  
 VYY = M22\*M41\*M41 + 2.0\*M41\*M42\*M21  
 RRY = M31\*M31\*M42 + 2.0\*M41\*M31\*M32  
 RYY = M32\*M41\*M41 + 2.0\*M41\*M42\*M31  
 PVR = M11\*M21\*M32+M31\*(M11\*M22+M12\*M21)  
 PVY = M11\*M21\*M42+M41\*(M11\*M22+M12\*M21)  
 PRY = M11\*M41\*M32+M31\*(M11\*M42+M12\*M41)  
 VRY = M21\*M41\*M32+M31\*(M21\*M42+M22\*M41)  
 PPP = 3.0\*M11\*M11\*M12  
 VVV = 3.0\*M21\*M21\*M22  
 RRR = 3.0\*M31\*M31\*M32  
 YYY = 3.0\*M41\*M41\*M42

C

L22=DPPV\*PPV+DPVV\*PVV+DPPR\*PPR+DPRR\*PRR+DPPY\*PPY+DPYY\*PYY  
 & +DVVR\*VVR+DVRR\*VRR+DVVY\*VVY+DVYY\*VYY+DRRY\*RRY+DRYY\*RYY  
 & +DPVR\*PVR+DPVY\*PVY+DPRY\*PRY+DVRY\*VRY+DPPP\*PPP+DVVV\*VVV  
 & +DRRR\*RRR+DYYY\*YYY

C

PPV = M12\*M12\*M21 + 2.0\*M11\*M12\*M22

PVV = M11\*M22\*M22 + 2.0\*M12\*M21\*M22  
 PPR = M12\*M12\*M31 + 2.0\*M11\*M12\*M32  
 PRR = M11\*M32\*M32 + 2.0\*M12\*M31\*M32  
 PPY = M12\*M12\*M41 + 2.0\*M11\*M12\*M42  
 PYY = M11\*M42\*M42 + 2.0\*M12\*M41\*M42  
 VVR = M22\*M22\*M31 + 2.0\*M21\*M22\*M32  
 VRR = M21\*M32\*M32 + 2.0\*M22\*M31\*M32  
 VVY = M22\*M22\*M41 + 2.0\*M21\*M22\*M42  
 VYY = M21\*M42\*M42 + 2.0\*M22\*M41\*M42  
 RRY = M32\*M32\*M41 + 2.0\*M31\*M32\*M42  
 RYY = M31\*M42\*M42 + 2.0\*M32\*M41\*M42  
 PVR = M12\*M22\*M31 + M32\*(M11\*M22+M12\*M21)  
 PVY = M12\*M22\*M41 + M42\*(M11\*M22+M12\*M21)  
 PRY = M12\*M42\*M31 + M32\*(M11\*M42+M12\*M41)  
 VRY = M22\*M42\*M31 + M32\*(M21\*M42+M22\*M41)  
 PPP = 3.0\*M11\*M12\*M12  
 VVV = 3.0\*M21\*M22\*M22  
 RRR = 3.0\*M31\*M32\*M32  
 YYY = 3.0\*M41\*M42\*M42

C

L23=DPPV\*PPV+DPVV\*PVV+DPPR\*PPR+DPRR\*PRR+DPPY\*PPY+DPYY\*PYY  
 & +DVVR\*VVR+DVRR\*VRR+DVVY\*VVY+DVYY\*VYY+DRRY\*RRY+DRYY\*RYY  
 & +DPVR\*PVR+DPVY\*PVY+DPRY\*PRY+DVRV\*VRY+DPPP\*PPP+DVVV\*VVV  
 & +DRRR\*RRR+DYYY\*YYY

C

L24= DPPV\*M12\*M12\*M22 + DPVV\*M12\*M22\*M22 + DPPR\*M12\*M12\*M32  
 & + DPRR\*M12\*M32\*M32 + DPPY\*M12\*M12\*M42 + DPYY\*M12\*M42\*M42  
 & + DVVR\*M22\*M22\*M32 + DVRR\*M22\*M32\*M32 + DVVY\*M22\*M22\*M42  
 & + DVYY\*M22\*M42\*M42 + DRRY\*M32\*M32\*M42 + DRYY\*M32\*M42\*M42  
 & + DPVR\*M12\*M22\*M32 + DPVY\*M12\*M22\*M42 + DPRY\*M12\*M42\*M32  
 & + DVRV\*M22\*M42\*M32 + DPPP\*M12\*M12\*M12 + DVVV\*M22\*M22\*M22  
 & + DRRR\*M32\*M32\*M32 + DYYY\*M42\*M42\*M42

C

L31=L21  
 L32=L22  
 L33=L23  
 L34=L24

C

L21=L21\*BB1\*U\*U  
 L22=L22\*BB1\*U\*U  
 L23=L23\*BB1\*U\*U  
 L24=L24\*BB1\*U\*U  
 L31=L31\*BB2\*U\*U  
 L32=L32\*BB2\*U\*U  
 L33=L33\*BB2\*U\*U  
 L34=L34\*BB2\*U\*U  
 L41=-0.5\*M11\*M11\*(M21+U\*M11/3.0)  
 L42=-M11\*(M12\*M21+0.5\*M11\*M22+0.5\*U\*M12\*M11)

C

$$L43 = -M12 * (M11 * M22 + 0.5 * M12 * M21 + 0.5 * U * M11 * M12)$$

$$L44 = -0.5 * M12 * M12 * (M22 + U * M12 / 3.0)$$

$$\begin{aligned} C11 &= (1/6.0) * YVVV * (M21 * M21 * M21) + 0.5 * YVRR * (M31 * M31 * M21) + \\ &\& 0.5 * YRVV * M21 * M21 * M31 + (1/6.0) * YPPP * M11 * M11 * M11 + (1/6.0) * YYYY * M41 * M41 * M41 \end{aligned}$$

$$\begin{aligned} C12 &= (1/6.0) * YVVV * (3 * M21 * M21 * M22) + 0.5 * YVRR * (M31 * M31 * \\ &\& M22 + 2 * M31 * M32 * M21) + 0.5 * YRVV * (M21 * M21 * M32 + 2 * M31 * M21 * M22) + (1/6.0) * YPPP * 3 * M11 * M11 * M12 \\ &\& + (1/6.0) * YYYY * 3 * M41 * M41 * M42 \end{aligned}$$

$$\begin{aligned} C13 &= (1/6.0) * YVVV * (3 * M21 * M22 * M22) + 0.5 * YVRR * (M21 * M32 * \\ &\& M32 + 2 * M31 * M32 * M22) + 0.5 * YRVV * (M22 * M22 * M31 + 2 * M32 * M21 * M22) + (1/6.0) * YPPP * 3 * M11 * M12 * M12 \\ &\& + (1/6.0) * YYYY * 3 * M41 * M42 * M42 \end{aligned}$$

$$\begin{aligned} C14 &= (1/6.0) * YVVV * (M22 * M22 * M22) + 0.5 * YVRR * (M32 * M32 * M22) + \\ &\& 0.5 * YRVV * M22 * M22 * M32 + (1/6.0) * YPPP * M12 * M12 * M12 + (1/6.0) * YYYY * M42 * M42 * M42 \end{aligned}$$

$$\begin{aligned} C21 &= (1/6.0) * NVVV * (M21 * M21 * M21) + 0.5 * (NVRR * M31 * M31 * M21) + \\ &\& 0.5 * NRVV * M21 * M21 * M31 + (1/6.0) * NPPP * M11 * M11 * M11 + (1/6.0) * NYYY * M41 * M41 * M41 \end{aligned}$$

$$\begin{aligned} C22 &= (1/6.0) * NVVV * (3 * M21 * M21 * M22) + 0.5 * NVRR * (M31 * M31 * \\ &\& M22 + 2 * M31 * M32 * M21) + 0.5 * NRVV * (M21 * M21 * M32 + 2 * M31 * M21 * M22) + (1/6.0) * NPPP * 3 * M11 * M11 * M12 \\ &\& + (1/6.0) * NYYY * 3 * M41 * M41 * M42 \end{aligned}$$

$$\begin{aligned} C23 &= (1/6.0) * NVVV * (3 * M21 * M22 * M22) + 0.5 * NVRR * (M21 * M32 * \\ &\& M32 + 2 * M31 * M32 * M22) + 0.5 * NRVV * (M22 * M22 * M31 + 2 * M32 * M21 * M22) + (1/6.0) * NPPP * 3 * M11 * M12 * M12 \\ &\& + (1/6.0) * NYYY * 3 * M41 * M42 * M42 \end{aligned}$$

$$\begin{aligned} C24 &= (1/6.0) * NVVV * (M22 * M22 * M22) + 0.5 * NVRR * M32 * M32 * M22 + \\ &\& 0.5 * NRVV * M22 * M22 * M32 + (1/6.0) * NPPP * M12 * M12 * M12 + (1/6.0) * NYYY * M42 * M42 * M42 \end{aligned}$$

$$D11 = (C11 * (IZ - NRDOT) + C21 * (YRDOT - M * XG)) / DVR$$

$$D12 = (C12 * (IZ - NRDOT) + C22 * (YRDOT - M * XG)) / DVR$$

$$D13 = (C13 * (IZ - NRDOT) + C23 * (YRDOT - M * XG)) / DVR$$

$$D14 = (C14 * (IZ - NRDOT) + C24 * (YRDOT - M * XG)) / DVR$$

$$D21 = (C11 * (NVDOT - M * XG) + C21 * (M - YVDOT)) / DVR$$

$$D22 = (C12 * (NVDOT - M * XG) + C22 * (M - YVDOT)) / DVR$$

$$D23 = (C13 * (NVDOT - M * XG) + C23 * (M - YVDOT)) / DVR$$

$$D24 = (C14 * (NVDOT - M * XG) + C24 * (M - YVDOT)) / DVR$$

C

C

C

Invert Transformation Matrix

DO 20 I=1,4

DO 30 J=1,4

```

        TINV(I,J)=0.0
        TSAVE(I,J)=T(I,J)
30      CONTINUE
20      CONTINUE
        CALL DLUD(4,4,TSAVE,4,TLUD,IVLUD)
        DO 40 I=1,4
            IF (IVLUD(I).EQ.0) STOP 3003
40      CONTINUE
        CALL DILU(4,4,TLUD,IVLUD,SVLUD)
        DO 80 I=1,4
            DO 90 J=1,4
                TINV(I,J)=TLUD(I,J)
90      CONTINUE
80      CONTINUE

C
C      Check Inv(T)*A*T
C
        IMULT=2
        IF (IMULT.EQ.1) CALL MULT(TINV,ASAVE,T)
        IF (IMULT.EQ.0) STOP

C
C      Definition of Nij
C
        N11=TINV(1,1)
        N21=TINV(2,1)
        N31=TINV(3,1)
        N41=TINV(4,1)
        N12=TINV(1,2)
        N22=TINV(2,2)
        N32=TINV(3,2)
        N42=TINV(4,2)
        N13=TINV(1,3)
        N23=TINV(2,3)
        N33=TINV(3,3)
        N43=TINV(4,3)
        N14=TINV(1,4)
        N24=TINV(2,4)
        N34=TINV(3,4)
        N44=TINV(4,4)

C
        R11=N12*L21+N13*L31+N14*L41+N12*D11+N13*D21
        R12=N12*L22+N13*L32+N14*L42+N12*D12+N13*D22
        R13=N12*L23+N13*L33+N14*L43+N12*D13+N13*D23
        R14=N12*L24+N13*L34+N14*L44+N12*D14+N13*D24
        R21=N22*L21+N23*L31+N24*L41+N22*D11+N23*D21
        R22=N22*L22+N23*L32+N24*L42+N22*D12+N23*D22
        R23=N22*L23+N23*L33+N24*L43+N22*D13+N23*D23
        R24=N22*L24+N23*L34+N24*L44+N22*D14+N23*D24

```

```

C
C      Evaluate Alpha' and Omega'
C
      DINC=0.001
      XDR =XD+DINC
      XDL =XD-DINC
      XD  =XDR

      CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
&      AA24,BB1,BB2,A,TL)

      CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
      CALL DSTABL(DEOS,WR,WI,FREQ)
      ALPHR=DEOS
      OMEGR=FREQ

      XD=XDL
C
      CALL LINEAR(K1,K3,XD,AA11,AA12,AA13,AA14,AA21,AA22,AA23,
&      AA24,BB1,BB2,A,TL)

      CALL RG(4,4,A,WR,WI,0,ZZZ,IV1,FV1,IERR)
      CALL DSTABL(DEOS,WR,WI,FREQ)
      ALPHL=DEOS
      OMEGL=FREQ
C
      DALPHA=(ALPHR-ALPHL)/(XDR-XDL)
      DOMEGA=(OMEGR-OMEGL)/(XDR-XDL)
C
C      Evaluation of Hopf Bifurcation Coefficients
C
      COEF1=3.0*R11+R13+R22+3.0*R24
      COEF2=3.0*R21+R23-R12-3.0*R14
      AMU2 =-COEF1/(8.0*DALPHA)
      BETA2=0.25*COEF1
      TAU2 =-(COEF2-DOMEGA*COEF1/DALPHA)/(8.0*OMEGAO)
      PER  =2.0*3.1415927/OMEGAO
      PER  =PER*U/L
C
      WRITE (15,2002) XD,WN,COEF1,DALPHA,PER

1 CONTINUE
C
1001 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF WN (dimensionless)')
1002 FORMAT (' ENTER MIN, MAX, AND INCREMENTS OF XD (dimensionless)')
1003 FORMAT (' ENTER DAMPING RATIO')

```

```

1006 FORMAT (' ENTER DSAT ')
1100 FORMAT (' ENTER TIME LAG TL (dimensional)')
2001 FORMAT (2I5)
2002 FORMAT (5E15.5)

```

END

C=====

```

      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
C
C      EVALUATES THE EIGENVALUE WITH THE MAXIMUM REAL PART
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(4),WI(4)
      DEOS=-1.0D+20
      DO 1 I=1,4
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
1      CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C
      SUBROUTINE LINEAR(K1,K2,XD,AA11,AA12,AA13,AA14,AA21,
& AA22,AA23,AA24,BB1,BB2,A,TL)
C
C      FORMS THE LINEARIZED MATRIX A (time delay 1st order approximation
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2
      DIMENSION A(4,4)
      A(1,1)=0.0D0
      A(1,2)=0.0D0
      A(1,3)=1.0D0
      A(1,4)=0.0D0
      A(2,1)=AA11+BB1*K1-BB1*K1*TL/XD
      A(2,2)=AA12-BB1*K1*TL/XD
      A(2,3)=AA13+BB1*K2
      A(2,4)=AA14+BB1*K1/XD
      A(3,1)=AA21+BB2*K1-BB2*K1*TL/XD
      A(3,2)=AA22-BB2*K1*TL/XD
      A(3,3)=AA23+BB2*K2
      A(3,4)=AA24+BB2*K1/XD
      A(4,1)=1.0D0
      A(4,2)=1.0D0

```

```

A(4,3)=0.0D0
A(4,4)=0.0D0
RETURN
END

```

C-----

```

SUBROUTINE DSOMEG(IJK,WR,WI,OMEGA,CHECK)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION WR(4),WI(4)
CHECK=-1.0D+25
DO 1 I=1,4
  IF (WR(I).LT.CHECK) GO TO 1
  CHECK=WR(I)
  IJ=I
1 CONTINUE
  OMEGA=DABS(WI(IJ))
  IF (WI(IJ).GT.0.0D0) IJK=IJ
  IF (WI(IJ).LT.0.0D0) IJK=IJ-1
  RETURN
END

```

C=====

C=====

```

SUBROUTINE NORMAL(T)
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION T(4,4),TNOR(4,4)
CFAC=T(1,1)**2+T(1,2)**2
IF (DABS(CFAC).LE.(1.D-10)) STOP 4001
TNOR(1,1)=1.D0
TNOR(2,1)=(T(1,1)*T(2,1)+T(2,2)*T(1,2))/CFAC
TNOR(3,1)=(T(1,1)*T(3,1)+T(3,2)*T(1,2))/CFAC
TNOR(4,1)=(T(1,1)*T(4,1)+T(4,2)*T(1,2))/CFAC
TNOR(1,2)=0.D0
TNOR(2,2)=(T(2,2)*T(1,1)-T(2,1)*T(1,2))/CFAC
TNOR(3,2)=(T(3,2)*T(1,1)-T(3,1)*T(1,2))/CFAC
TNOR(4,2)=(T(4,2)*T(1,1)-T(4,1)*T(1,2))/CFAC
DO 1 I=1,4
  DO 2 J=1,2
    T(I,J)=TNOR(I,J)
2 CONTINUE
1 CONTINUE
RETURN

```

```

      END
C=====
      SUBROUTINE MULT(TINV,A,T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION TINV(4,4),A(4,4),T(4,4),A1(4,4),A2(4,4)
      DO 1 I=1,4
        DO 2 J=1,4
          A1(I,J)=0.D0
          A2(I,J)=0.D0
2      CONTINUE
1     CONTINUE
      DO 3 I=1,4
        DO 4 J=1,4
          DO 5 K=1,4
            A1(I,J)=A(I,K)*T(K,J)+A1(I,J)
5          CONTINUE
4        CONTINUE
3      CONTINUE
      DO 6 I=1,4
        DO 7 J=1,4
          DO 8 K=1,4
            A2(I,J)=TINV(I,K)*A1(K,J)+A2(I,J)
8          CONTINUE
7        CONTINUE
6      CONTINUE
      DO 11 I=1,4
C      WRITE (*,101) (A(I,J),J=1,4)
11     CONTINUE
      DO 12 I=1,4
C      WRITE (*,101) (T(I,J),J=1,4)
12     CONTINUE
      DO 10 I=1,4
        WRITE (*,101) (A2(I,J),J=1,4)
10     CONTINUE
C      WRITE (*,101) A2(1,1)
      RETURN
101  FORMAT (4E15.5)
      END

```



## LIST OF REFERENCES

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